Problem for 1996 March

Proposed by Professor Stu Smith

For \( n=1,2,3,\ldots \) let \( X_n \) be the continued fraction as follows.

\[
X_n = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots + \frac{1}{a_n}}}}
\]

Show that when \( X_n \) is written as a "proper fraction" (i.e., no fraction in numerator or denominator) the number of terms in both the numerator and the denominator are Fibonacci numbers (i.e., elements of the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots).

For example,

\[
X_4 = \frac{a_1 a_2 a_3 a_4 + a_1 a_2 + a_1 a_4 + a_3 a_4 + 1}{a_2 a_3 a_4 + a_2 + a_4}
\]

in which the numerator consists of 5 terms and the denominator consists of 3 terms.