Problem for 1996 April

Proposed by Dan Jurca

Prove that if \( m \) and \( n \) are integers, \( 0 \leq m \), and \( 1 \leq n \), then \((n!)^m! | (mn)!\).

Remarks.

i. The symbol `|' means `divides'.

ii. You might like to compute the quotient (which is, of course, asserted to be an integer).

iii. The proposer recently came across a reference to this result in a Canadian mathematics magazine (the *Crux Mathematicorum*), in which it was claimed to be ``well known''. Since it was not well known to the proposer, the proposer wanted to prove it is true.

Solution by the proposer

*Proposition:* \( m,n \in \mathbb{Z}, 0 \leq m, 1 \leq n \Rightarrow (n!)^m! | (mn)!\).

Proof.

The assertion clearly holds in the case \( m=0 \) and \( 1 \leq n \), so assume from now on that \( 1 \leq m \) and \( 1 \leq n \).

**Lemma 1:** \( n,x \in \mathbb{Z}, 1 \leq n \leq x \Rightarrow x(x-1)(x-2)\ldots[x-(n-1)]=n!(x \parallel n)\).

Proof of lemma 1.

\[
x(x-1)(x-2)\ldots[x-(n-1)] = \frac{x!}{((x-n)!)!}=n!\left(\frac{x!}{(n!(x-n)!)}\right)=n!(x \parallel n),
\]

proving lemma 1.

Remark: Hence \( n! \) divides the product of any \( n \) consecutive integers.

**Lemma 2:** \( m,n \in \mathbb{Z}, 1 \leq m, 1 \leq n \Rightarrow (mn \parallel n)=m((mn-1) \parallel (n-1))\).

Proof of lemma 2.
We complete the proof by expressing \((mn)!\), the product of \(mn\) factors, as a product of \(m\) products, each of \(n\) consecutive integer factors, as follows:

\[
(mn)! = (mn) \cdot (mn-1) \cdot (mn-2) \cdot \ldots \cdot (2) \cdot (1)
\]

\[
= \prod_{k=1}^{m} \left\{ (m+1-k)n \cdot (m+1-k)n-1 \cdot \ldots \cdot (m+1-k)n-(n-1) \right\}
\]

\[
= \prod_{k=1}^{m} n! \left( \frac{(m+1-k)n}{n} \right), \text{ by lemma 1}
\]

\[
= (n!)^m \prod_{k=1}^{m} \left( \frac{(m+1-k)n}{n} \right)
\]

\[
= (n!)^m \prod_{k=1}^{m} (m+1-k) \left( \frac{(m+1-k)n-1}{n-1} \right), \text{ by lemma 2}
\]

\[
= (n!)^m m! \prod_{k=1}^{m} \left( \frac{(m+1-k)n-1}{n-1} \right).
\]

The last line above exhibits the integral quotient, completing the proof.

Also solved by Eric Leong.