Problem for 1996 September and October

Proposed by Dan Jurca

Suppose that the rational function (quotient of polynomials) \( p(x)/q(x) \) nicely approximates the function \( e^x \) for small \( x \), in the sense that

\[
\frac{p(x)}{q(x)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^N}{N!} + O(x^{N+1}).
\]

Assume that

\[
p(x) = \sum_{i=0}^{m} a_i x^i, \quad q(x) = \sum_{i=0}^{n} b_i x^i,
\]

and we define

\[
-\ p(x) = \sum_{i=0}^{m} -a_i x^i, \quad -\ q(x) = \sum_{i=0}^{n} -b_i x^i,
\]

where

\[
- a_i = \begin{cases} 
  a_i & \text{for } i \text{ even;} \\
  -a_i & \text{for } i \text{ odd;}
\end{cases} \quad \text{and} \quad - b_i = \begin{cases} 
  b_i & \text{for } i \text{ even;} \\
  -b_i & \text{for } i \text{ odd.}
\end{cases}
\]

Show that \([([ -q(x)])/([ -p(x)])] \) approximates \( e^x \) equally well; i.e.,

\[
- q(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^N}{N!} + O(x^{N+1}).
\]
For example, we find

\[
\frac{720+600x+240x^2+60x^3+10x^4+x^5}{720-120x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + O(x^7);
\]

\[
\frac{720+120x}{720-600x+240x^2-60x^3+10x^4-x^5} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + O(x^7).
\]

Solution by the proposer

We remark that the rational approximation of \(e^x\) given in the problem is an instance of a Padé approximation; and in the special case of the exponential function the Padé approximation of bidegree \((m,n)\) is explicitly known [1, page 24]:

\[
e^x \approx \frac{\sum_{i=0}^{m} \binom{m}{i} (m+n-i)! x^i}{\sum_{i=0}^{n} \binom{n}{i} (m+n-i)! (-x)^i}
\]

from which the assertion of the problem follows at once. We can prove a somewhat more general result:

**Proposition.** Suppose \(\phi\) is an analytic function in some neighborhood of 0 and for \(x\) in that neighborhood

\[
\phi(x) = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \left[ \frac{1}{\epsilon} - \epsilon \right] x^{n+1} + \ldots
\]
Then with \( \psi(x) = \frac{1}{\phi(-x)} \) we have, in a neighborhood of 0,

\[
\psi(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \left[ \frac{1}{(n+1)!} - (-1)^n e \right] x^{n+1} + \ldots
\]

Before proving this proposition, we remark that there are indeed such approximations of \( e^x \) which are not rational functions, for example (the silly)

\[
e^x \approx \sqrt{1-x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \ldots = 1 + x + \frac{x^2}{2!} + \left[ -\frac{1}{3!} \right] x^3 + \ldots
\]

Also the proposition gives the assertion of the problem, where

\[
\phi(x) = \frac{p(x)}{q(x)}, \text{ and } \psi(x) = \frac{q(-x)}{p(-x)} = \frac{1}{\phi(-x)}.
\]

Proof of proposition. First we observe the well-known

\[
1 \leq i \Rightarrow \sum_{j=0}^{i} (-1)^j \binom{i}{j} = \sum_{j=0}^{i} \binom{i}{j} i^{i-j} (-1)^j = [1 + (-1)]^{i} = 0. \quad (*)
\]

Then we recall Leibniz’s formula, easily proved by induction:

Suppose for \( 0 \leq i \) that there exist \( i \) derivatives of \( u \) and of \( v \); then there exist \( i \) derivatives of the product \( uv \), and moreover \( (uv)^{(k)} = \sum_{j=0}^{i} \binom{i}{j} u^{(i-j)} v^{(j)} \) \( (**) \)

where \( ^{(k)} \) denotes \( k \)-th derivative.

By hypothesis \( \phi(-x) \psi(x) = 1 \), at least for \( x \) near 0. Clearly \( 0 \leq i \leq n \Rightarrow \phi^{(i)}(0) = 1 \); we shall show that the same holds for \( \psi^{(i)}(0) \).
Observe first that from $\phi(0)\psi(0)=1$ we have $\psi(0)=1$; assume (inductively) $1 \leq i \leq n$, and $0 \leq j < i \Rightarrow \psi^{(j)}(0)=1$. By the chain rule $0 \leq i \Rightarrow [\phi(-x)]^{(i)}=(-1)^i\phi^{(i)}(-x)$.

By (**) 

\[
1 \leq i \leq n \Rightarrow [\phi(-x)\psi(x)]^{(i)}=1^{(i)}=0 = \sum_{j=0}^{i} \binom{i}{j} [\phi(-x)]^{(i-j)}\psi^{(j)}(x)
\]

\[
= \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} \phi^{(i-j)}(-x)\psi^{(j)}(x).
\]

Then with $x=0$ we find, since $\phi^{(i-1)}(0)=1$, that $\sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} \psi^{(j)}(0)=0$.

Hence

\[
1 \leq i \leq n \Rightarrow \psi^{(i)}(0) = (-1)^{i-1} \sum_{j=0}^{i-1} (-1)^{j} \binom{i}{j} \psi^{(j)}(0)
\]

\[
= (-1)^{i-1} \sum_{j=0}^{i-1} (-1)^{j} \binom{i}{j} \text{ by inductive hypothesis}
\]

\[
= (-1)^{i-1}.(-1)^{i-1} \text{ by (*)}
\]

\[
=1.
\]

Thus

\[
0 \leq i \leq n \Rightarrow \psi^{(i)}(0)=1.
\]

(***)

By (**) again we have

\[
\sum_{j=0}^{n+1} \binom{n+1}{j} [\phi(-x)]^{(n+1-j)}\psi^{(j)}(x)=0,
\]
whence

\[
\psi^{(n+1)}(0) = -(-1)^{n+1}\phi^{(n+1)}(0) - \sum_{j=1}^{n} \binom{n+1}{j} (-1)^{n+1-j} \\
= (-1)^n\phi^{(n+1)}(0) - (-1)^{n+1} \sum_{j=1}^{n} (-1)^j \binom{n+1}{j} \\
= (-1)^n\phi^{(n+1)}(0) - (-1)^{n+1} \cdot [-1 - (-1)^{n+1}] \\
= (-1)^n\phi^{(n+1)}(0) + (-1)^{n+1} + 1.
\]

from which one easily completes the proof of the proposition.