Problem for 1996 November

Proposed by Dan Jurca

Suppose that \( n \) is a non-negative integer, \( r \in \mathbb{R} \), and \( 1 < |r| \); then (by the ratio test) \( \sum_{i=1}^{\infty} \left[ \frac{(i^n)}{(r^i)} \right] \) converges; find a formula, or practical method, to evaluate the sum precisely.

Solution by the proposer

For a fixed \( r \in \mathbb{R} \), \( 1 < |r| \) we write, for non-negative integers \( n \): \( S_n=\sum_{i=1}^{\infty} \left[ \frac{(i^n)}{(r^i)} \right] \). Then

\[
S_0=\sum_{i=1}^{\infty} \frac{1}{r^i} = \frac{1/r}{1-1/r} = \frac{1}{r-1}.
\]

Now suppose \( 1 \leq n \); we shall compute \( S_n \) in terms of \( S_0, S_1, \ldots, S_{n-1} \) as follows.

\[
S_n=\sum_{i=1}^{\infty} \frac{i^n}{r^i}
\]

so that

\[
rS_n = \sum_{i=1}^{\infty} \frac{i^n}{r^{i-1}}
= \sum_{i=0}^{\infty} \frac{(i+1)^n}{r^i}
= \sum_{i=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} i^j \frac{1}{r^i}
\]
\[
\begin{align*}
\sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{\infty} r^i j^i &= \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{\infty} r^i j^i \\
&= \sum_{j=0}^{n} \binom{n}{j} S_j \\
&= \sum_{j=0}^{n-1} \binom{n}{j} S_j + S_n.
\end{align*}
\]

Therefore

\[
(r-1)S_n = \sum_{i=0}^{n-1} \binom{n}{i} S_i,
\]

whence

\[
S_n = \frac{1}{r-1} \sum_{i=0}^{n-1} \binom{n}{i} S_i,
\]

from which each $S_n$ may be computed recursively.