Problem for 1997 February

Proposed by Lishan Chen and Dan Jurca

An array \([a_1, \ldots, a_n]\) of, let us say, integers is called a heap if:

\[
1 \leq i \leq \lfloor n/2 \rfloor \Rightarrow a_{2i} \leq a_i \quad \text{and} \quad 1 \leq i \leq \lfloor (n-1)/2 \rfloor \Rightarrow a_{2i+1} \leq a_i.
\]

(One thinks of a heap as a binary tree with \(a_1\) in the root node; \(a_2\) in the left child node of the root node, \(a_3\) in the right child node of the root node; \(a_4\) and \(a_5\) in the child nodes of the node containing \(a_2\), etc. Then a heap is a complete binary tree in which each parent is at least as big as her children.)

Consider a positive integer \(n\), and all \(n!\) arrays consisting of data from \(\{1, 2, 3, \ldots, n\}\) with no repetitions. Some of these arrays are heaps. (For example, if \(n=6\), then the array \([6, 4, 5, 2, 1, 3]\) is a heap but \([5, 6, 1, 2, 3, 4]\) is not a heap.) How many of these \(n!\) arrays are heaps? Find a method to answer this question. In particular, how many arrays of length 10 (consisting of the data from \(\{1, 2, 3, \ldots, 10\}\) with no repetitions) are heaps? How about the case \(n=100\)?

Solution by the proposers

Let \(h(n)\) be the number of heaps which can be made from a set of \(n\) distinct real numbers. We shall give a recursive formula for \(h(n)\). First we recall that a heap is conveniently represented as a complete binary tree-a binary tree with \(2^i\) nodes on level \(i\) (the root node is on level 0), except possibly the bottom level; and all the bottom-level nodes are as far to the left as possible. For each positive integer \(n\) there is a unique shape for a complete binary tree with \(n\) nodes; let \(l(n)\) be the number of nodes in the left subtree of the root node of such a tree, and let \(r(n)\) be the number of nodes in the right subtree. Then writing \(n=2^h+m\) where \(h\) is a nonnegative integer and \(0 \leq m < 2^h\) (\(h\) and \(m\) are unique), one has

\[
l(n) = \begin{cases} 
2^{h-1} + m & \text{if } m < 2^{h-1}, \\
2^h - 1 & \text{if } 2^{h-1} \leq m.
\end{cases}
\]

(For example, one finds easily \(l(10)=6; l(100)=63\.) Obviously, \(r(n)=n-1-l(n)\).

Now it is immediate that if the array \([a_1, a_2, \ldots, a_n]\) is a heap (consisting of \(n\) distinct real numbers),
then $a_1$ is the greatest element of the array, and if the array is arranged in the form of a complete binary tree, then the left subtree of the root node (which contains $a_1$) and the right subtree of the root node are also heaps. Clearly the elements in the left subtree can be chosen in \((n-1) \parallel l(n)\) ways; the remaining \(r(n)\) elements are in the right subtree. Hence we have the following result:

\[
\begin{aligned}
h(n) &= \begin{cases} 
1 & \text{if } n \leq 2; \\
\binom{n-1}{l(n)} \cdot h(l(n)) \cdot h(r(n)) & \text{if } 2 < n.
\end{cases}
\end{aligned}
\]

It is now a simple matter to compute, for example, \(h(6)=20\), and \(h(10)=3,360\).

Although we do not have a formula for \(h(n)\) in closed form, we can make the following remarks:

i. If \(2 < n\), then either the left subtree (of a complete binary tree with \(n\) nodes) or the right subtree is \textit{full}; \textit{i.e.}, has \(2^i\) nodes on level \(i\) for \(i=0,1,2,\ldots,h\), where \(h\) is the \textit{height} (maximum level) of the tree. Since a full binary tree with height \(h\) has exactly \(2^{h+1}-1\) nodes, and since (for \(0 \leq n, 0 \leq m \leq n\) \((n \parallel m) = (n \parallel (n-m))\), it is possible to express the binomial coefficient in the above recursive formula for \(h(n)\) so that the "bottom number" is 1 less than a power of 2.

ii. One can always "reduce" the evaluation of \(h(n)\) to evaluation of a product of binomial coefficients (with "bottom number" 1 less than a power of 2) and factors \(h(2^k-1)\) for various \(k\). For example, we find (since \(l(100)=63\) and \(r(100)=36\), etc.)

\[
h(100) = \binom{99}{63} \cdot h(63)h(36)
\]

\[
= \binom{99}{63} \cdot h(35) \cdot \binom{15}{10} \cdot h(20)h(15)
\]

\[
= \binom{99}{63} \cdot \binom{35}{15} \cdot h(63)h(15) \cdot \binom{19}{12} \cdot h(12)h(7)
\]

\[
= \binom{99}{63} \cdot \binom{35}{15} \cdot \binom{19}{7} \cdot h(63)h(15)h(7) \cdot \binom{11}{7} \cdot h(7)h(4)
\]

\[
= \binom{99}{63} \cdot \binom{35}{15} \cdot \binom{19}{7} \cdot \binom{11}{7} \cdot h(63)h(15)h(7)^2 \cdot \binom{3}{2} \cdot h(2)h(1)
\]

\[
= \binom{99}{35} \cdot \binom{19}{7} \cdot \binom{11}{3} \cdot h(63)h(15)h(7)^2.
\]
iii. We do have a formula for $h$ of such $n$; it follows at once (and is trivially proved by induction) from the recursive formula given above that

$$1 \leq k \Rightarrow h(2^{k-1}) = \prod_{i=1}^{k-1} \left( \frac{2(2^{k-i}-1)}{2^{k-i}-1} \right)^{2^{i-1}}.$$ 

For example, we find

$$h(63) = \left( \frac{62}{31} \right) \left( \frac{30}{15} \right) \left( \frac{14}{7} \right)^2 \left( \frac{6}{3} \right)^4 \left( \frac{2}{1} \right)^{16};$$

$$h(15) = \left( \frac{14}{7} \right) \left( \frac{6}{3} \right)^2 \left( \frac{2}{1} \right)^4;$$

$$h(7) = \left( \frac{6}{3} \right)^2 \left( \frac{2}{1} \right)^2.$$ 

Therefore, continuing with the above example,

$$h(100) = \left( \frac{99}{63} \right) \left( \frac{35}{15} \right) \left( \frac{19}{7} \right) \left( \frac{11}{7} \right) \left( \frac{3}{1} \right) \left( \frac{62}{31} \right) \left( \frac{30}{15} \right) \left( \frac{14}{7} \right)^5 \left( \frac{6}{3} \right)^{12} \left( \frac{2}{1} \right)^{24},$$

evaluation of which product we leave for the reader. \(\square\)