Problem for 1997 September and October

Let $\mathbb{Q}$ be the set of rational numbers, so $\mathbb{Q} \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Let $\mathbb{R}^2$ be topologized as usual, and consider $S=(\mathbb{Q} \times (\mathbb{R} - \mathbb{Q})) \cup ((\mathbb{R} - \mathbb{Q}) \times \mathbb{Q}) \subseteq \mathbb{R}^2$. Thus $S$ is the set of points in the Cartesian plane with one coordinate rational and the other coordinate irrational.

Prove that $S$ is not connected.

Solution by Professor Emeritus Victor Manjarrez

The line with equation $y=x$ is disjoint from $S$, and is a closed subset of $\mathbb{R}^2$; hence let $U=\{(x,y) \in \mathbb{R}^2 \mid x < y\}$ and $V=\{(x,y) \in \mathbb{R}^2 \mid y < x\}$. Then $U$ is open in $\mathbb{R}^2$, $V$ is open in $\mathbb{R}^2$, neither $U$ nor $V$ is empty, $U \cap V = \emptyset$, and $S \subseteq U \cup V$.

Therefore $S$ is not connected.