Problem for 1999 February

Proposed by Professor Emeritus Victor Manjarrez

Prove that if $A$ is a finite subset of $\mathbb{C}$, the set of complex numbers, and if

$$|| \sum_{a \in A} a || = \sum_{a \in A} |a|,$$

then for each subset $B$ of $A$

$$|| \sum_{b \in B} b || = \sum_{b \in B} |b|.$$

Solution by the proposer

If $C = A \setminus B$, then

$$\sum_{a \in A} |a| = || \sum_{a \in A} a ||$$

$$= || \sum_{b \in B} b + \sum_{c \in C} c ||$$

$$\leq || \sum_{b \in B} b || + || \sum_{c \in C} c ||$$

$$\leq \sum_{b \in B} |b| + \sum_{c \in C} |c|$$

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\[ = \sum_{a \in A} |a|. \]

It follows that all these are equal, so that

\[ || \sum_{b \in B} b || + || \sum_{c \in C} c || = \sum_{b \in B} |b| + || \sum_{c \in C} c || , \]

whence \[ |\sum_{b \in B} b| = \sum_{b \in B} |b|, \] as asserted.

Remark: The assertion and proof are immediately generalized to normed spaces.