The n-simplex \( \triangle^n \) may be defined as follows.

\[
\triangle^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leq i \leq n \Rightarrow 0 \leq x_i, \text{ and } x_0 + x_1 + \ldots + x_n = 1 \}
\]

The sketch below shows \( \triangle^0 = A, \triangle^1 = AB, \triangle^2 = ABC; \triangle^3 \) is a tetrahedron in \( \mathbb{R}^4 \).

Let us say that the 0-volume of a finite set of points is the number of points in the set, the 1-volume of a line segment is the length of the segment, the 2-volume of a polygonal region is the area of the region, the 3-volume of a polyhedron is the ordinary volume, etc.

Find a formula for the n-volume of \( \triangle^n \).

Solution by the proposer

Writing \( V_n \) for the n-volume of \( \triangle^n \) we show by induction on \( n \) that \( 0 \leq n \Rightarrow V_n = \sqrt{(n+1)/n!} \). This clearly holds if \( n=0 \) (for \( \triangle^0 \) is a set consisting of a single point), or \( n=1 \) (for \( \triangle^1 \) is a line segment of length \( \sqrt{2} \)), or \( n=2 \) (for \( \triangle^2 \) is the closed region bounded by an equilateral triangle of side \( \sqrt{2} \)). So suppose that \( 2 \leq n \) and \( V_{n-1} = \sqrt{n/(n-1)!} \). The following figure shows that we may consider \( \triangle^n \) as an n-dimensional "pyramid" with base \( \triangle^{n-1} \) and height, say, \( h_n \). (The figure shows \( \triangle^3 \) as a tetrahedron with each side of length \( \sqrt{2} \), base \( \triangle^2 \) an equilateral triangle with sides of length \( \sqrt{2} \), and height \( h_3 \)). Now \( \triangle^n \) may be considered made up of slices which look like \( \triangle^{n-1} \); the \((n-1)\)-volume of such a slice at distance \( x \) from the apex of \( \triangle^n \) is equal to \( (x/h_n)^{n-1} \times V_{n-1} \). (This clearly holds if \( x=0 \) or \( x=h_n \); for \( 0 < x < h_n \) it holds almost by definition of \((n-1)\)-volume-the \((n-1)\)-volume of an \((n-1)\)-dimensional cube is the \((n-1)\)-th power of the length of an edge of the cube.) Therefore we have

\[
1 \leq n \Rightarrow V_n = \int_{h_n}^x (x - \_\_\_\_\_\_\_)^{n-1} V_{n-1} \text{thinspace dx}
\]
\[
\begin{align*}
\mathbf{0} & \quad (h_n) \\
= & \quad \frac{1}{n} h_n V_{n-1}.
\end{align*}
\]

To complete the argument by induction we need to determine \( h_n \). Now \( h_n \) is the distance from the point \((0,0,\ldots,0,1) \in \mathbb{R}^{n+1}\) to \( \triangle^{n-1} \), which may clearly be assumed to be a subset of \( \mathbb{R}^{n+1} \) as well as of \( \mathbb{R}^n \). Thus \( h_n \) is the distance from \((0,0,\ldots,0,1) \) to \( \triangle^{n-1} = \{(x_0,x_1,\ldots,x_{n-1},0) \mid 0 \leq x_i \text{ and } x_0+x_1+\ldots+x_{n-1}=1\} \). We may compute this distance by minimizing the square of the distance from the apex, \((0,0,\ldots,0,1)\), to the points in \( \triangle^{n-1} \). So consider

\[
\begin{align*}
f(x_0,x_1,\ldots,x_{n-2}) & = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 + 1 \\
& = x_0^2 + x_1^2 + \ldots + (1-x_0-x_1-\ldots-x_{n-2})^2 + 1.
\end{align*}
\]

Computing the partial derivatives of \( f \) with respect to the \( x_i \) we have

\[
\frac{\partial f}{\partial x_i} (x_0,x_1,\ldots,x_{n-2}) = 2x_i + 2(1-x_0-x_1-\ldots-x_{n-2}) - 1 \\
= 2x_i - 2 + 2x_0 + 2x_1 + \ldots + 2x_{n-2},
\]

so setting each of these \( n-1 \) partial derivatives to zero yields the following \((n-1)\times(n-1)\) linear system.

\[
\begin{bmatrix}
2 & 1 & 1 & \ldots & 1 \\
1 & 2 & 1 & \ldots & 1 \\
1 & 1 & 2 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 2 \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{n-2}
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]

The coefficient matrix, say \( A \), above is nonsingular. In fact the determinant equals \( n \). This may be verified by observing that \( A \) has the following \( n-1 \) linearly independent eigenvectors.
\( \mathbf{v}_1 = (1, 1, 1, \ldots, 1)^T \)
\( \mathbf{v}_2 = (1, -1, 0, 0, \ldots, 0)^T \)
\( \mathbf{v}_3 = (1, 0, -1, 0, \ldots, 0)^T \)
\( \mathbf{v}_4 = (1, 0, 0, -1, 0, \ldots, 0)^T \)
\[
\vdots
\]
\( \mathbf{v}_{n-1} = (1, 0, 0, 0, \ldots, 0, -1)^T, \)

with associated eigenvalues \( n, 1, 1, 1, \ldots, 1 \). Therefore there exists a unique solution of the linear system; by inspection it is \( x_i = 1/n \), for \( i = 0, 1, \ldots, n-2 \). It follows that \( h_n \), the distance from the apex \((0, 0, \ldots, 0, 1)\) of \( \triangle_n \) to its base \( \triangle_{n-1} \), is equal to the distance in \( \mathbb{R}^{n+1} \) from the point \((0, 0, \ldots, 0, 1)\) to the point \((1/n, 1/n, \ldots, 1/n, 0)\), which equals

\[
h_n = \sqrt{(1/n)^2 + (1/n)^2 + \ldots + (1/n)^2 + 1}
\]

\[
= \sqrt{1/n + 1}
\]

\[
= \sqrt{n+1} / n.
\]

We can now complete the inductive argument.

\[
V_n = \frac{1}{n} h_n V_{n-1}
\]

\[
= \frac{1}{n} \times \sqrt{n+1} / n \times \sqrt{n} / (n-1)!
\]

\[
= \sqrt{n+1} / n!
\]
as asserted.

It may be interesting to observe that \((V_n)_{n=0}^\infty \to 0\) quite rapidly, and \((h_n)_{n=1}^\infty \to 1\).

Also solved by John M. Sayer