Problem for 2002 March

Proposed by Dan Jurca

For $0 \leq n$ let $H_n$ be the sum of the first $n$ terms of the harmonic series; i.e.,

$$H_n = \begin{cases} 
0 & \text{if } n=0; \\
\sum_{i=1}^{n} \frac{1}{i} & \text{if } 1 \leq n.
\end{cases}$$

Find a formula for

$$\sum_{i=0}^{n} iH_i.$$ 

The proposer encountered this sum when analyzing a certain variation of the quicksort algorithm.

Solution by the proposer

Proposition.
\[ 0 \leq n \Rightarrow \sum_{i=0}^{n} iH_i = \frac{n^2+n}{2} H_n - \frac{n^2-n}{4} \]

Proof.

The formula obviously holds if \( n = 0 \); now suppose that \( 1 \leq n \) and

\[ \sum_{i=0}^{n-1} iH_i = \frac{(n-1)^2+(n-1)}{2} H_{n-1} - \frac{(n-1)^2-(n-1)}{4} \]

\[ = \frac{n^2-2n+1+n-1}{2} H_{n-1} - \frac{n^2-2n+1-n+1}{4} \]

\[ = \frac{n^2-n}{2} H_{n-1} - \frac{n-1}{n} \frac{n^2-3n+2}{4} \]

\[ = \frac{n^2-n}{2} H_n - \frac{n^2-n}{4} H_{n-1} - \frac{n^2-3n+2}{4} \]

\[ = \frac{n^2-n}{2} H_n - \frac{n^2-n}{4} \]

Then

\[ \sum_{i=0}^{n} iH_i = \frac{n^2-n}{2} H_n + nH_n - \frac{n^2-n}{4} \]

\[ = \frac{n^2+n}{2} H_n - \frac{n^2-n}{4} \]

so the proposition follows by induction on \( n \).