Problem for 2002 April

Proposed by Dan Jurca

Find a formula for the following sum, where n is a nonnegative integer.

\[ \sum_{1 \leq i < j \leq n} ij \]

Solution by the proposer

We show that the sum equals \([(n(3n+2)(n^2-1))/24]\). For with

\[ S_n = \sum_{1 \leq i < j \leq n} ij, \quad \text{we have} \]

\[
S_n = \begin{cases} 
0 & \text{if } n=0, \\
\sum_{1 \leq i < j \leq n-1} ij + \left( \sum_{i=1}^{n-1} i \right) \times n & \text{if } 1 \leq n; \text{ so that} \\
S_{n-1} + \frac{(n-1)n^2}{2} & \text{if } 1 \leq n. 
\end{cases} 
\]
Now the asserted formula obviously holds if $n=0$; suppose that $1 \leq n$ and

\[
S_{n-1} = \frac{(n-1)[3(n-1)+2][(n-1)^2-1]}{24}; \quad \text{then}
\]

\[
S_n = S_{n-1} + \frac{(n-1)n^2}{2}
\]

\[
= \frac{(n-1)(3n-1)(n^2-2n)+12(n-1)n^2}{24}
\]

\[
= \frac{(n-1)n[(3n-1)(n-2)+12n]}{24}
\]

\[
= \frac{(n-1)n[3n^2-7n+2+12n]}{24}
\]

\[
= \frac{(n-1)n(3n^2+5n+2)}{24}
\]

\[
= \frac{(n-1)n(n+1)(3n+2)}{24}
\]

\[
= \frac{n(3n+2)(n^2-1)}{24}
\]

so that by induction the formula holds for each nonnegative integer $n$. 