Problem for 2002 June

Communicated by Dan Jurca

This is an old problem, but has not appeared here for a long time.

There are one million dots (points) in the plane; prove that there exists a (straight) line such that exactly half of the dots are on each side of the line.

Solution (from a book of problems by Charles Trigg)

We generalize to the case that there are $2n$ dots, where $n$ is a positive integer. Let us call this set of points $S$. Consider the set of all

$$\binom{2n}{2} = n(2n-1)$$

(not necessarily distinct) lines determined by each pair of points chosen from $S$. These lines do not cover the entire plane—the plane is not the union of finitely many lines. Hence there exists a point, say $P$, which does not lie on any of the lines, and furthermore lies to the left of some disk the interior of which includes $S$. Now for each point $Q$ in $S$ the line determined by $P$ and $Q$ contains no point of $S$ other than $Q$. Therefore as the point $Q$ varies through $S$ we may mark the $n$-th and the $(n+1)$-th lines; each line through $P$ between these lines has $n$ points of $S$ on each side.

Also solved by Matthew Hubbard and John Sayer.