Problem for 2002 November

Communicated by Matthew Hubbard

Find a polynomial $p(x,y)$ such that

$$(x,y) \in \mathbb{R}^2 \Rightarrow 0 < p(x,y)$$

and

$$\inf \{ p(x,y) \mid (x,y) \in \mathbb{R}^2 \} = 0.$$ 

Solution by Dan Jurca

Consider the following polynomial, $p$.

$$p(x,y) = x^2y^2 + y^2 + 2xy + 1 = y^2 + (xy + 1)^2$$

Obviously $\forall (x,y) \in \mathbb{R}^2; \ 0 \leq p(x,y)$; and clearly $p(x,y)=0$ if and only if both $y=0$ and $xy+1=0$, which is impossible, so that

$$(x,y) \in \mathbb{R}^2 \Rightarrow 0 < p(x,y).$$

Next, if $0 < \varepsilon$, then
so that \( \text{im } p = p(\mathbb{R}^2) = \{ t \in \mathbb{R} | 0 < t \} \), and therefore

\[
\inf \{ p(x,y) | (x,y) \in \mathbb{R}^2 \} = 0.
\]