Problem for 2002 December

Communicated by Dan Jurca

The following problem appears on page 59 of *A Course of Modern Analysis* by E.T. Whittaker and G.N. Watson.

Show that the series

\[ \sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \]

is equal to \([ 1/((1-z)^2) ]\) when \(|z| < 1\) and is equal to \([ 1/(z(1-z)^2) ]\) when \(1 < |z|\).

Solution by Dan Jurca

First suppose \(|z| \neq 1\). We shall show

\[ 1 \leq N \Rightarrow \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1-z^N}{(1-z)^2(1-z^{N+1})}. \]

(We take \(0^0=1\) here.) First suppose \(N=1\). Then
\[ \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{z^{1-1}}{(1-z)(1-z^{1+1})} \]

\[ = \frac{1}{(1-z)(1-z^2)} \]

\[ = \frac{1-z}{(1-z)^2(1-z^2)} \cdot \]

so the assertion certainly holds if N=1. Next suppose 2 \leq N and

\[ \sum_{n=1}^{N-1} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1-z^{N-1}}{(1-z)^2(1-z^N)} \cdot \]

Then

\[ \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1-z^{N-1}}{(1-z)^2(1-z^N)} + \frac{z^{N-1}}{(1-z^N)(1-z^{N+1})} \]

\[ = \frac{(1-z^{N-1})(1-z^{N+1})+(1-z^2)z^{N-1}}{(1-z)^2(1-z^N)(1-z^{N+1})} \]

\[ = \frac{1-z^{N-1}-z^{N+1}+z^{2N}+z^{N-1}-2z^N+z^{N+1}}{(1-z)^2(1-z^N)(1-z^{N+1})} \]

\[ = \frac{(1-z^N)^2}{(1-z)^2(1-z^N)(1-z^{N+1})} \cdot \]

so that the assertion holds for each N, 1 \leq N, by induction on N. Therefore, if |z| < 1, then \( z^N \to 0 \) as \( N \to \infty \), whence

\[ \sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \]

\[ = \frac{1-z^N}{1-z} \]
\[
\lim_{N \to \infty} (1-z)^2 (1-z^{N+1}) = \frac{1}{(1-z)^2}.
\]

Next suppose \(1 < |z|\). Then with \(w=1/z\) we have \(|w| < 1\), so that

\[
\sum_{n=1}^{\infty} \frac{w^{n-1}}{(1-w^n)(1-w^{n+1})} = \frac{1}{(1-w)^2}.
\]

Thus, replacing \(w\) with \(1/z\) and negating each factor in the denominators, as we may,

\[
\sum_{n=1}^{\infty} \frac{1/z^{n-1}}{(1/z^n-1)(1/z^{n+1}-1)} = \frac{1}{(1/z-1)^2}, \quad \text{so}
\]

\[
\sum_{n=1}^{\infty} \frac{z^n z^{n+1}/z^n}{(1-z^n)(1-z^{n+1})} = \frac{z^2}{(1-z)^2}, \quad \text{so}
\]

\[
\sum_{n=1}^{\infty} \frac{z^{n+2}}{(1-z^n)(1-z^{n+1})} = \frac{z^2}{(1-z)^2}, \quad \text{so}
\]

\[
z^3 \sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{z^2}{(1-z)^2}, \quad \text{so that, finally,}
\]

\[
\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{z(1-z)^2},
\]

as desired.
Also solved by Farid El-Mouchrif and John M. Sayer