Problem for 2003 January

Proposed by Dan Jurca

Simplify the following expression.

\[
\sqrt[3]{\frac{4018030018}{5}} + \sqrt[5]{1614456522554908032} + \sqrt[3]{\frac{4018030018}{5}} - \sqrt[5]{1614456522554908032}
\]

Solution by the proposer

Observing that \(1614456522554908032^5 = 4018030018^2 + 1\) and writing \(x\) for the value of the given expression, we have

\[
x = \sqrt[3]{u + \sqrt{u^2 + 1}} + \sqrt[3]{u - \sqrt{u^2 + 1}}
\]

where \(u = 4018030018\). Then from \((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\), we have

\[
x^3 = u + \sqrt{u^2 + 1} + 3 \sqrt[3]{(u + \sqrt{u^2 + 1})^2(u - \sqrt{u^2 + 1})}
\]

\[
+3 \sqrt[3]{(u + \sqrt{u^2 + 1})(u - \sqrt{u^2 + 1})^2 + u - \sqrt{u^2 + 1}}
\]
\[=2u+3 \sqrt[3]{(2u^2+1+2u \sqrt{u^2+1})(u- \sqrt{u^2+1})} +
\]
\[3 \sqrt[3]{(u+ \sqrt{u^2+1})(2u^2+1-2u \sqrt{u^2+1})}
\]
\[=\ldots
\]
\[=2u+3 \left[ \sqrt[3]{-u+ \sqrt{u^2+1}} + \sqrt[3]{-u+ \sqrt{u^2+1}} \right]
\]
\[=2u-3 \left[ \sqrt[3]{u+ \sqrt{u^2+1}} + \sqrt[3]{u- \sqrt{u^2+1}} \right]
\]
\[=2u-3x.
\]
Therefore the real number \(x\) satisfies the equation \(x^3+3x-2u=0\). Since the function \(f: \mathbb{R} \to \mathbb{R}\) by \(f(t)=t^3+3t-2u\) strictly increases on \(\mathbb{R}\) (\(0 < 3t^2+3=f'(t))\), the equation has a unique real root. When \(u=4018030018\) one finds the unique real root equals 2003; therefore the given expression simplifies to the number 2003.

Also solved by Farid El-Mouchrif and Dangthu Ta