Problem for 2003 June

Consider the Fibonacci sequence

\[(f_n)_{n=0}^{\infty} = (0,1,1,2,3,5,8,13,21,\ldots).\]

That is,

\[f_n = \begin{cases} 
0 & \text{if } n=0 \\
1 & \text{if } n=1 \\
 f_{n-2}+f_{n-1} & \text{if } 2 \leq n.
\end{cases}\]

Then let

\[F(x) = \sum_{n=0}^{\infty} f_n x^n = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \ldots.\]

a. For which \(x\) does the series above converge?

b. Find a formula for the function \(F(x)\).

c. Compute \(F(1/2)\).

d. Sketch the graph of \(F\).
Solution by the proposer

Let \( \varphi = (1+\sqrt{5})/2 \) and observe that \(-1/\varphi = (1-\sqrt{5})/2\). We recall the following formula.

\[
0 \leq n \Rightarrow f_n = (\varphi^n - (-1/\varphi)^n)/\sqrt{5}
\]

Now obviously \( 1 < \varphi \), so that by the root test the series above converges if \(|x| < 1/\varphi\) and diverges if \(1/\varphi < |x|\). Next we observe that (since \(0 < \varphi\)) \(f_n/\varphi^n = 1/\sqrt{5}[1-(\varphi^{-2n})]\) does not converge to 0 as \(n \to \infty\); similarly, \(f_n/(-\varphi)^n = 1/\sqrt{5}[(-1)^n - 1/\varphi^{2n}]\) does not converge to 0 as \(n \to \infty\); thus the interval of convergence of the series is precisely \((-1/\varphi, 1/\varphi)\). We compute as follows.

\[
\begin{align*}
xF(x) &= f_0x + f_1x^2 + f_2x^3 + f_3x^4 + \ldots \\
x^2F(x) &= f_0x^2 + f_1x^3 + f_2x^4 + \ldots \\
\text{thus} \quad xF(x) + x^2F(x) &= f_0x + f_2x^2 + f_3x^3 + f_4x^4 + \ldots \\
&= F(x) - x; \\
\text{hence} \quad F(x) &= \frac{x}{1-x-x^2}.
\end{align*}
\]

(One might find it amusing to actually carry out this division.)

Therefore \(F(1/2) = [(1/2)/(1-1/2-1/4)] = 2\).

Also solved by Walt Becker, James Farrell, Massoud Malek, and John Sayer