Problem for 2003 July

Communicated by Dan Jurca

Recall that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is additive if

$$\forall x, y \in \mathbb{R}: \varphi(x+y) = \varphi(x) + \varphi(y).$$

The following problem appears on page 36 of *Conjecture and Proof* by Miklós Laczkovich.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

for every $x, y \in \mathbb{R}$. Prove that there is a constant $c$ such that $f-c$ is additive.

Solution by Dan Jurca

Let $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = f(x) - f(0)$. We claim that $x \in \mathbb{R} \Rightarrow \varphi(2x) = 2\varphi(x)$. For

$$\varphi(x) = \varphi\left(\frac{2x+0}{2}\right)$$

$$= f\left(\frac{2x+0}{2}\right) - f(0)$$

$$= \frac{f(2x) + f(0)}{2} - f(0)$$

$$= \frac{f(2x) - f(0) + f(0) - f(0)}{2}$$

$$= f(2x) - f(0)$$
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\[ \frac{f(2x) - f(0)}{2} = \frac{\phi(2x)}{2}, \]
so \( \phi(2x) = 2\phi(x). \)

But then \( x, y \in \mathbb{R} \Rightarrow \)

\[
\phi(x+y) = \phi\left(\frac{2x+2y}{2}\right) = f\left(\frac{2x+2y}{2}\right) - f(0) = \frac{f(2x) + f(2y) - f(0)}{2} = \frac{\phi(2x) + \phi(2y)}{2} = \frac{2\phi(x) + 2\phi(y)}{2} = \phi(x) + \phi(y).
\]

Therefore with \( c = f(0) \) the function \( f-c \) is additive.

Also solved by Kirk Demlinger, James Farrell, Kurt Luoto, and Massoud Malek