Problem for 2004 April

Communicated by Dan Jurca

According to the 2004 March issue of the Canadian mathematics journal *Crux Mathematicorum* the following problem appeared in the 2003 Croatian Mathematical Society National Competition, Junior Level.

Prove that if the product of the positive real numbers $x$, $y$, and $z$ is equal to 1, and

$$x+y+z \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

then for each nonnegative integer $n$

$$x^n+y^n+z^n \leq \frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n}.$$ 

Solution by Dan Jurca

Assume $0 < x \leq y \leq z$, so that $x \leq 1$, else $1 < xyz$. Since $xyz=1$, we have $z=1/(xy)$; next from

$$x+y+\frac{1}{xy} \leq \frac{1}{x} + \frac{1}{y} + xy$$  we have

$$x^2y+xy^2+1 \leq y+x+x^2y^2,$$  so

$$0 \leq (x^2-x)y^2-(x^2-1)y+(x-1)$$  whence

$$0 \leq (x-1)[xy^2-(x+1)y+1].$$

Now if $x=1$, then $yz=1$ so $z=1/y$ and the inequality to be shown becomes simply $y^n+(1/y)^n \leq 1/(y^n)+y^n$, which is obvious; hence assume $x < 1$. Now $xy^2-(x+1)y+1=0$ iff $y=1$ or $y=1/x$, so since $xy^2-(x+1)y+1 \leq 0$, we deduce that $1 \leq y \leq 1/x$. 

Now suppose $\alpha$ is a positive real number, let $S=\{(u,v) \in \mathbb{R}^2 \mid 0 < u \leq 1 \text{ and } 1 \leq v \leq 1/u\}$, and consider $f:S \to \mathbb{R}$ by $f(u,v)=u^\alpha+v^\alpha+1/(uv)^\alpha-1/u^\alpha-1/v^\alpha-(uv)^\alpha$. We observe that $(x,y) \in S$ and

$$f(u,1)=0,$$

$$f \left( u, \frac{1}{u} \right)=0; \text{ and}$$

$$1 < v \Rightarrow \lim_{u \to 0^+} f(u,v)=-\infty, \text{ and}$$

$$\frac{\partial f}{\partial u}(u,v) = \alpha u^{\alpha-1} - u \frac{1}{u^{\alpha+1} v^{\alpha}} + \alpha \frac{1}{u^{\alpha+1} v^{\alpha}} - \alpha u^{\alpha-1} v^\alpha$$

$$= \frac{\alpha(1-v^\alpha)}{u^{\alpha+1} v^{\alpha}} [(uv)^\alpha u^{\alpha-1}],$$

and since $1-v^\alpha \leq 0$ and $(uv)^\alpha u^{\alpha-1} \leq 0$, it follows that $0 \leq \partial f/\partial u$, so that for each $v_0$, $1 < v_0$, the function $g:(0,1/v_0) \to \mathbb{R}$ by $g(u)=f(u,v_0)$ increases from $-\infty$ at $0^+$ to $0$ at $1/v_0$, so we have $f(u,v) \leq 0$ for each $(u,v) \in S$. Hence (with $u=x$ and $v=y$)

$$x^\alpha+y^\alpha+\frac{1}{(xy)^\alpha} \leq \frac{1}{x^\alpha} + \frac{1}{y^\alpha}+(xy)^\alpha,$$

and the desired inequality holds for each nonnegative real number $n$. 