Problem for 2004 September and October

Proposed by Dan Jurca

For each positive integer \( n \) define \( S(n) \) as follows.

\[
S(n) = \sum_{b=2}^{n} \log_{b} n
\]

\[= \log_{2} n + \log_{3} n + \log_{4} n + \ldots + \log_{n} n\]

Determine whether

\[
\lim_{n \to \infty} \frac{S(n)}{n}
\]

exists; and if it exists, determine its value.

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Solution by the proposer

We have

\[
S(n) = \sum_{b=2}^{n} \log_{b} n = \sum_{b=2}^{n} \frac{\ln n}{\ln b} = \ln n \sum_{b=2}^{n} \frac{1}{\ln b}.
\]

The function \((1, \infty) \to \mathbb{R}\) by \( t \to \frac{1}{\ln t} \) assumes only positive values and strictly decreases, so

\[
1 < x \Rightarrow \frac{1}{\ln(x+1)} < \int_{x+1}^{x+1} \frac{dt}{\ln t} < \frac{1}{\ln x};
\]
hence (by summing)

\[ 3 \leq n \Rightarrow \int_2^n \frac{dt}{\ln t} + \int_n^{n+1} \frac{dt}{\ln t} < \sum_{b=2}^{n} \frac{1}{\ln b} = \frac{1}{\ln 2} + \sum_{b=3}^{n} \frac{1}{\ln b} < \frac{1}{\ln 2} + \sum_{b=3}^{n} \int_{b-1}^{b} \frac{dt}{\ln t} \]

= \frac{1}{\ln 2} + \int_2^n \frac{dt}{\ln t}

whence

\[ 3 \leq n \Rightarrow \frac{\ln n}{n} \cdot \int_2^n \frac{dt}{\ln t} < S(n) < \frac{\ln n}{n} \cdot \frac{1}{\ln 2} + \frac{\ln n}{n} \cdot \int_2^n \frac{dt}{\ln t}. \]

Now

\[ \lim_{x \to \infty} \frac{\ln x}{x} = 0 \text{ and } \int_2^\infty \frac{dt}{\ln t} = \infty \text{ (since } 1 < t \Rightarrow \frac{1}{t} < \frac{1}{\ln t} \text{)} \]

so by l'Hôpital's rule and the fundamental theorem of calculus

\[ \lim_{x \to \infty} \frac{\ln x}{x} \cdot \int_2^x \frac{dt}{\ln t} = 1; \]

therefore by the squeeze theorem
\[
\lim_{n \to \infty} S(n) = 1.
\]

Also solved by Dennis Eichhorn and Sarah Frey