Problem for 2006 February

Proposed by Dan Jurca

According to the 2005 December issue of the Canadian mathematics journal *Crux Mathematicorum* the following problem appeared in the 2005 Maritime Mathematics Competition.

Three students play a game with the understanding that the loser is to double the money of each of the other two. After three games, each has lost once and each has $24. How much did each student have to start?

a. Solve the problem above.
b. Generalize as follows. Suppose there are $n$ players, say the players are $P_1, P_2, \ldots, P_n$, and the notation is such that player $P_i$ loses exactly once, the $i$-th game. Again suppose that the loser of a game doubles the money of each of the other players, that the $n$ players play $n$ games, and that after these $n$ games each player has an amount $A$. For $1 \leq i \leq n$ and $0 \leq j \leq n$ let $a_{ij}$ be the amount had by player $P_i$ after $j$ games have been played. Thus player $P_i$ begins with $a_{i0}$, and for each $i$, $1 \leq i \leq n$, $a_{in}=A$. Determine $a_{ij}$.

Solution by Dan Jurca

By the conditions in the statement of the problem one has the following.

$$
1 \leq i \leq n \text{ and } 1 \leq j \leq n \Rightarrow a_{ij} = \begin{cases} 
2a_{i,j-1} & \text{if } j \neq i, \\
\sum_{k=1, k \neq i}^{n} a_{k,j-1} & \text{if } i=j.
\end{cases}
$$

Since after $n$ games the total amount of money had by all the players equals $nA$, it follows that

$$
0 \leq j \leq n \Rightarrow \sum_{k=1}^{n} a_{kj} = nA, \text{ so that }
$$
\[0 \leq j \leq n \Rightarrow \sum_{k=1, k \neq i}^{n} a_{kj} = nA - a_{ij},\]

and one can restate the condition above as follows.

\[
1 \leq i \leq n \text{ and } 1 \leq j \leq n \Rightarrow a_{ij} = \begin{cases} 
2a_{i,j-1} & \text{if } j \neq i, \\
2a_{i,i-1} - nA & \text{if } i = j.
\end{cases}
\]

Now there exists a \(n \times (n+1)\) matrix with rows 1, 2, \ldots, \(n\) and columns 0, 1, \ldots, \(n\) containing \(a_{ij}\) in row \(i\) and column \(j\) as follows.

\[
\begin{pmatrix}
a_{10} & 2a_{10} - nA & 4a_{10} - 2nA & 8a_{10} - 4nA & \ldots & 2^n a_{10} - 2^{n-1} nA \\
a_{20} & 2a_{20} & 4a_{20} - nA & 8a_{20} - 2nA & \ldots & 2^n a_{20} - 2^{n-2} nA \\
a_{30} & 2a_{30} & 4a_{30} & 8a_{30} - nA & \ldots & 2^n a_{30} - 2^{n-3} nA \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n0} & 2a_{n0} & 4a_{n0} & 8a_{n0} & \ldots & 2^n a_{n0} - nA
\end{pmatrix}
\]

It follows by inspection that

\[
1 \leq i \leq n \text{ and } 0 \leq j \leq n \Rightarrow a_{ij} = \begin{cases} 
2^i a_{i0} & \text{if } j < i, \\
2^i a_{i0} - 2^i nA & \text{if } i \leq j.
\end{cases}
\]
Since $1 \leq i \leq n \Rightarrow a_{ii}=A$, we find

$$2^n a_{i0} - 2^{n-i} n A = A, \text{ whence }$$

$$a_{i0} = \frac{A + 2^{n-i} n A}{2^n}$$

$$= \frac{2^{n-i} n + 1}{2^n} A, \text{ so that, finally }$$

$$1 \leq i \leq n \text{ and } 0 \leq j \leq n \Rightarrow a_{ij} = \left\{ \begin{array}{ll}
2^{n-i+j} n + 2^j & \text{ if } j < i, \\
2^n & \text{ if } i \leq j,
\end{array} \right.$$

which one can verify by induction.

In the special case $n=3$, $A=24$, we have $a_{10}=$39, $a_{20}=$21, and $a_{30}=$12.

Also solved by Massoud Malek