Problem for 2006 April

Proposed by Dan Jurca

The following problem is a variation of problem 11209 which appears in the 2006 March issue of The American Mathematical Monthly.

Show that there do not exist positive real numbers \(x_1, x_2,\) and \(x_3\) which satisfy the following system of equations.

\[
\begin{align*}
(x_1+x_2+x_3)^{x_1} &= \frac{6}{7} \\
(x_1+x_2+x_3)^{x_2} &= \frac{7}{8} \\
(x_1+x_2+x_3)^{x_3} &= \frac{8}{9}
\end{align*}
\]

Solution by the proposer

Assuming that there exist \(x_1, x_2,\) and \(x_3\) which satisfy the given system we obtain a contradiction as follows. With \(X=x_1+x_2+x_3\) we find the product of the equations yields the equation

\[
X^{x_1}X^{x_2}X^{x_3} = \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9}, \quad \text{so that}
\]

\[
X^{x_1+x_2+x_3} = \frac{6}{9}, \text{or}
\]

\[
X^X = \frac{2}{3}.
\]

However, if \(\varphi:(0,\infty)\to\mathbb{R}\) by \(\varphi(x)=x^x\), we find

\[
\varphi'(x) = x^x(1+\ln x)
\]

\[
= 0 \text{ if and only if } x=\frac{1}{e},
\]

and that \(\varphi_{\min} = \varphi(\frac{1}{e}) = (\frac{1}{e})^{\frac{1}{e}}\). Since \(2/3 < (\frac{1}{e})^{\frac{1}{e}} \approx 0.6922\), there exists no real \(X\) such that \(X^X = \frac{2}{3}\); hence there exists no solution of the system.
Also solved by Massoud Malek, Craig Prescott, John M. Sayer, and Nathan Speed