Problem for 2006 May

Communicated by Dan Jurca

a. Show that there exists a bijection $\mathbb{R}^\mathbb{N} \to \mathbb{R}$; i.e., from the set of sequences of real numbers to the set of real numbers. 
Here $\mathbb{N} = \{0, 1, 2, \ldots \}$, the set of natural numbers, and $\mathbb{R}$ is the set of real numbers.

b. Deduce that if $I$ is a non-empty real interval, then $C(I)$, the set of continuous functions $I \to \mathbb{R}$, has the cardinality of $\mathbb{R}$.
Remark. The proof shows more generally that if $X$ is a non-empty separable Hausdorff space, then the cardinality of $C(X)$ equals the cardinality of $\mathbb{R}$.

Solution by Dan Jurca

We recall the definitions: for sets $A$ and $B$ the set $B^A$ equals the set of all functions from $A$ to $B$, and a sequence in a set $S$ is an element of $S^\mathbb{N}$; i.e., a sequence in $S$ is a function $\mathbb{N} \to S$. One way to verify the existence of a bijection $\mathbb{R}^\mathbb{N} \to \mathbb{R}$ is to recall that there exists a bijection $\mathbb{R} \to 2^{\mathbb{N}} = \{0, 1\}^\mathbb{N}$, where $2^{\mathbb{N}} = \{0, 1\}^\mathbb{N}$ is the set of binary sequences-sequences of 0s and 1s. (A nice argument is in chapter 2 of Notes on Set Theory by Yiannis Moschovakis.) Now a function $\varphi : B \to C$ induces $\varphi^* : B^A \to C^A$ by $f \mapsto \varphi^f$, and it is trivial to verify that if $\varphi : B \to C$ is a bijection with inverse $\psi : C \to B$, then $\psi^* : C^A \to B^A$ is the inverse of $\varphi^*$. So that $\varphi^*$ is also a bijection. This suffices to show that there exists a bijection $\langle \{0, 1\}^\mathbb{N} \rangle^\mathbb{N} \to \{0, 1\}^\mathbb{N}$; i.e., there exists a bijection from the set of sequences of binary sequences to the set of binary sequences. We can do this using Cantor's ``first diagonal method''.

So suppose $X \in \langle \{0, 1\}^\mathbb{N} \rangle^\mathbb{N}$; say $X = (x_i)_{i=0}^\infty$, where each $x_i$ is a binary sequence, say $x_i = (x_{i,j})_{j=0}^\infty$, where each $x_{i,j} \in \{0, 1\}$. We consider the doubly infinite array as follows.

$$
\begin{align*}
\mathbf{x}_0 &= (x_{0,0}, x_{0,1}, x_{0,2}, x_{0,3}, \ldots) \\
\mathbf{x}_1 &= (x_{1,0}, x_{1,1}, x_{1,2}, x_{1,3}, \ldots) \\
\mathbf{x}_2 &= (x_{2,0}, x_{2,1}, x_{2,2}, x_{2,3}, \ldots) \\
\mathbf{x}_3 &= (x_{3,0}, x_{3,1}, x_{3,2}, x_{3,3}, \ldots) \\
&\vdots
\end{align*}
$$

We associate with this sequence $X$ of binary sequences the binary sequence $y$, where
\[ y = (x_{0,0}, x_{0,1}, x_{1,0}, x_{0,2}, x_{1,1}, x_{2,0}, x_{0,3}, x_{1,2}, x_{2,1}, x_{3,0}, \ldots). \]

It is easy to see that \( y \) is well-defined, and that \( X \) can be recovered from \( y \); \textit{i.e.}, \((\{0,1\}^N)^N \rightarrow \{0,1\}^N\) by \( \rightarrow y \) is a bijection. Thus there exists a bijection \( \mathbb{R}^N \rightarrow \mathbb{R} \).

Also solved by Bill Nico