Problem for 2006 August

Proposed by Dan Jurca

Prove the following.

$$\lim_{n \to \infty} \int_{0}^{\pi/2} \sin^n \theta \, d\theta = 0$$

Solution by the proposer

Suppose $0 < \varepsilon < \pi/2$. Let $a=(\pi-\varepsilon)/2$, so $0 < a < \pi/2$. Now

$$\int_{0}^{\pi/2} \sin^n \theta \, d\theta = \int_{0}^{a} \sin^n \theta \, d\theta + \int_{a}^{\pi/2} \sin^n \theta \, d\theta.$$  

Since $0 < \sin a < 1$, $\sin^n a \to 0$ as $n \to \infty$; hence there exists $n_0$ such that $1 \leq n_0$ and $n_0 \leq n \Rightarrow \sin^n a \leq \varepsilon/(2a)$, so $n_0 \leq n \Rightarrow a\sin a \leq \varepsilon/2$. Therefore, since $1 \leq n \Rightarrow \sin^n$ increases on the interval $[0,a]$,

$$n_0 \leq n \Rightarrow \int_{0}^{a} \sin^n \theta \, d\theta < (a-0)\sin^n a$$

$$\leq \varepsilon/2.$$  

Also

$$\int_{a}^{\pi/2} \sin^n \theta \, d\theta < (\pi/2-a)\sin^n(\pi/2)$$

$$=(\pi/2-a) \times 1$$

$$=\varepsilon/2.$$
Therefore

\[ n_0 \leq n \Rightarrow \int_0^{\pi/2} \sin^n \theta \, d\theta < \varepsilon, \text{ whence} \]

\[ \lim_{n \to \infty} \int_0^{\pi/2} \sin^n \theta \, d\theta = 0. \]

Also solved by Massoud Malek and John M. Sayer