Problem for 2006 December

Proposed by Dan Jurca

The following is a slight variation on Mathematical Mayhem problem 213, which appeared (with solution) in the 2006 November edition of the Canadian mathematics journal *Crux Mathematicorum*.

Suppose x and y are (real or complex) numbers, and P equals the following product of n factors.

\[
P = (x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)(x^{16}+y^{16})\ldots
\]

Find the value of P.

Solution by the proposer

First suppose x=y; then

\[
P = 2x \times 2x^2 \times 2x^4 \times \ldots \times 2x^{2n-1}
\]

\[
= 2^n x^{2+4+\ldots+2n-1}
\]

\[
= 2^n x^{2n-1}.
\]

Next

\[
(x-y)P = (x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)\ldots(x^{2n-1}+y^{2n-1})
\]

\[
= (x^2-y^2)(x^2+y^2)(x^4+y^4)(x^8+y^8)\ldots(x^{2n-1}+y^{2n-1})
\]

\[
= (x^4-y^4)(x^4+y^4)(x^8+y^8)\ldots(x^{2n-1}+y^{2n-1})
\]

\[
= (x^8-y^8)(x^8+y^8)\ldots(x^{2n-1}+y^{2n-1})
\]

\[
\vdots
\]

\[
= x^{2n} - y^{2n}.
\]

Hence
\[ P = \begin{cases} 
2^n x^{2n-1} & \text{if } x = y, \\
\frac{x^{2n} - y^{2n}}{x - y} & \text{if } x \neq y.
\end{cases} \]

Remark. One observes that if \( f(y) = y^{2n} - x^{2n} \), then

\[
\lim_{y \to x} \frac{x^{2n} - y^{2n}}{x - y} = \lim_{y \to x} \frac{y^{2n} - x^{2n}}{y - x} = f'(x) = 2^n x^{2n-1}.
\]

Also solved by Grant Morgan