Problem for 2007 February

Proposed by Dan Jurca

For each \( n \geq 1 \), let

\[
q_n = \frac{2 \times 5 \times 8 \times 11 \times 14 \times \cdots \times (3n - 1)}{1 \times 4 \times 7 \times 10 \times 13 \times \cdots \times (3n - 2)}.
\]

Determine whether

\[
\lim_{n \to \infty} q_n
\]

exists, and, if the limit exists, its value.

Solution by the proposer

The limit does not exist (or equals \( \infty \)), as shown here. We have

\[
\ln q_n = \ln \frac{2 \times 5 \times 8 \times \cdots \times (3n - 1)}{1 \times 4 \times 7 \times \cdots \times (3n - 2)}
= \ln \frac{2}{1} + \ln \frac{5}{4} + \cdots + \ln \frac{3n - 1}{3n - 2}
= \sum_{i=1}^{n} \ln \frac{3i - 1}{3i - 2}
= \sum_{i=1}^{n} \ln \left(1 + \frac{1}{3i - 2}\right).
\]

From the well-known (and easily proved) inequality

\[-1 < x \Rightarrow \frac{x}{1 + x} < \ln(1 + x)\]

we have

\[
1 \leq i \Rightarrow \frac{1}{3i - 2} < \ln \left(1 + \frac{1}{3i - 2}\right), \quad \text{so that}
\]

\[
1 \leq i \Rightarrow \frac{1}{3i - 1} < \ln \left(1 + \frac{1}{3i - 2}\right), \quad \text{whence}
\]

\[
1 \leq i \Rightarrow \frac{1}{3i} < \ln \left(1 + \frac{1}{3i - 2}\right).
\]

Therefore

\[
1 \leq n \Rightarrow \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \frac{1}{3} H_n < \ln q_n,
\]

where \( H_n = 1 + 1/2 + 1/3 + \cdots + 1/n \), and since \( H_n \to \infty \) as \( n \to \infty \), it follows that \( q_n \to \infty \) as well.

Also solved by Kelly Hubble, Massoud Malek, and Nathan Speed