Problem for 2008 January

Communicated by Dan Jurca

Prove that if
\[ a \in \mathbb{R}, \ b \in \mathbb{R}, \ c \in \mathbb{R}; \]
\[ a < b < c; \]
\[ a + b + c = 2; \quad \text{and} \]
\[ ab + bc + ca = 1; \]
then
\[ 0 < a < 1/3; \]
\[ 1/3 < b < 1; \quad \text{and} \]
\[ 1 < c < 4/3. \]

Solution by Dan Jurca

Since \( c = 2 - a - b \) we have from \( ab + bc + ca = 1 \) that \( ab + b(2-a-b) + (2-a-b)a = 1 \), so that \( a^2 + b^2 + ab - 2a - 2b + 1 = 0 \), or \( b^2 + (a-2)b + (a^2 - 2a + 1) = 0 \). Therefore \( b = \frac{(-a + 2 + \sqrt{a^2 - 4a + 4 - 4a^2 + 8a - 4})}{2} = (-a + 2 + \sqrt{4a - 3a^2})/2 \). Since \( b < c = 2 - a - b \) we have \( 2b < 2 - a \), so that if \( b = -a + 2 + \sqrt{4a - 3a^2}/2 \), then \( 2b = -a + 2 + \sqrt{4a - 3a^2} < 2 - a \), so that \( 4a - 3a^2 < 0 \), which is impossible. Therefore \( b = (-a + 2 - \sqrt{4a - 3a^2})/2 \). Now \( 3a^2 < 4a \), so if \( a < 0 \) then \( 4 \leq 3a \), whence \( 4/3 \leq a \), which is contradictory; therefore \( 0 \leq a \). But if \( a = 0 \), then \( b = 1 \), so \( 1 < c \), contradicting \( a + b + c = 2 \). Since \( a < b \) we have \( 2a < -a + 2 - \sqrt{4a - 3a^2} \), so \( 3a - 2 < -\sqrt{4a - 3a^2} \). Since the function \( x \mapsto x^2 \) decreases in the interval \((-\infty, 0]\) it follows that \( 4a - 3a^2 < 9a^2 - 12a + 4 \), whence \( 0 < 12a^2 - 16a + 4 \) so \( 0 < 3a^2 - 4a + 1 = (3a - 1)(a - 1) \). Hence either \( a < 1/3 \) or \( 1/3 < a < 1 \). Since \( a \leq 1 \) (from \( 0 < a < b < c \) and \( a + b + c = 2 \) we must have \( a < 1/3 \). Thus \( 0 < a < 1/3 \). Now consider \( \varphi : [0, 1/3] \to \mathbb{R} \) by \( \varphi(x) = (-x + 2 - \sqrt{4x - 3x^2})/2 \). We find \( \varphi(0) = 1, \ \varphi(1/3) = 1/3 \), and
\[
\varphi'(x) = \frac{1}{2} \left( -1 + \frac{2 - 3x}{\sqrt{4x - 3x^2}} \right)
\]
\[
= \frac{1}{2} \frac{\sqrt{4x - 3x^2} + 2 - 3x}{\sqrt{4x - 3x^2}}
\]
\[
< 0
\]
so that \( \varphi \) decreases; hence, since \( b = \varphi(a) \), it follows that \( 1/3 < b < 1 \).

Next, \( c = 2 - a - b = 2 - a - (2 - a - \sqrt{4a - 3a^2})/2 = (2 - a + \sqrt{4a - 3a^2})/2 \). With \( \psi : [0, 1/3] \to \mathbb{R} \) by \( \psi(x) = (2 - x + \sqrt{4x - 3x^2})/2 \) we find that \( \psi(0) = 1, \ \psi(1/3) = 4/3 \), and if \( 0 < x < 1/3 \) then
\[
\psi'(x) = \frac{1}{2} \left( -1 + \frac{2 - 3x}{\sqrt{4x - 3x^2}} \right)
\]
\[
= \frac{2 - 3x - \sqrt{4x - 3x^2}}{2\sqrt{4x - 3x^2}}.
\]

Now with \( \theta : [0, 1/3] \to \mathbb{R} \) by \( \theta(x) = 4x - 3x^2 \) we find \( \theta(0) = 0, \ \theta(1/3) = 1 \), and \( \theta \) increases (since \( 0 < \theta' \)), so that \( \theta_{max} = \theta(1/3) = 1/3 \). From this it follows that \( 0 < \psi' \), so \( \psi \) increases in \([0, 1/3]\), and finally, since \( c = \psi(a) \), \( 1 < c < 4/3 \).

Also solved by Massoud Malek and Grant Morgan