Problem for 2008 July

Proposed by Dan Jurca

Prove that for integers \( m \) and \( n \) with \( 1 \leq m < n \)

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = 0.
\]

Solution by the proposer

Let

\[
S(m, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m;
\]

we first show that

\[2 \leq m \text{ and } 2 \leq n \Rightarrow S(m, n) = n[S(m-1, n) - S(m-1, n-1)].\]

For

\[
S(m-1, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^{m-1}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (-1)^i \binom{n}{i} i^{m-1} \quad \text{and}
\]

\[
S(m-1, n-1) = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} i^{m-1}
\]
\[ i = 0 \]
\[ = \frac{1}{n} \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} i^m \]
\[ = \frac{1}{n} \sum_{i=1}^{n} (-1)^i \binom{n-i}{i} i^m, \text{ whence} \]
\[ S(m-1,n) - S(m-1,n-1) = \frac{1}{n} \sum_{i=1}^{n} (-1)^i \left( \frac{n}{i} - \frac{n-i}{i} \right) \binom{n}{i} i^m \]
\[ = \frac{1}{n} \sum_{i=1}^{n} (-1)^i \binom{n}{i} i^m \]
\[ = \frac{1}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m \]
\[ = \frac{1}{n} S(m,n), \]

from which the result follows.

Next we claim \( 1 < n \Rightarrow S(1,n)=0 \). For \( 1 < n \Rightarrow \)

\[ S(1,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i \]
\[ = \sum_{i=1}^{n} (-1)^i \binom{n}{i} \]
\[ = n \sum_{i=1}^{n} (-1)^i \binom{n-1}{i-1} \]
\[
\sum_{i=1}^{n} \left( \sum_{i=1}^{n-1} \binom{n-1}{i-1} i^{(n-1)-(i-1)} (-1)^{i-1} \right)
=\sum_{i=0}^{n-1} \binom{n-1}{i} 1^{(n-1)-i} (-1)^i
=-(1-1)^{n-1}
=0.
\]

We now have

\[2 < n \Rightarrow S(2,n) = n[S(1,n)-S(1,n-1)]=0, \quad (\text{since } 1 < n-1)\]
\[3 < n \Rightarrow S(3,n) = n[S(2,n)-S(2,n-1)]=0, \quad (\text{since } 2 < n-1)\]

\text{etc., so that } 1 \leq m < n \Rightarrow S(m,n)=0.

Remark. In fact \((-1)^n S(m,n)\) equals the number of surjections from an \(m\)-element set to an \(n\)-element set; obviously this number equals zero if \(m < n\).

Also solved by Bojan Basic (Serbia), Minghua Lin (China), and John M. Sayer