Problem for 2008 November

Communicated by Dan Jurca

Prove: \(1 \leq n \Rightarrow \sum_{k=1}^{n} \left[ \frac{(-1)^{k-1}}{k} \right] (n \parallel k) = \sum_{k=1}^{n} \frac{1}{k}. \)

Solution by Dan Jurca

Consider the polynomial function \(f_n: \mathbb{R} \to \mathbb{R}\) by

\[
f_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} x^{k-1}
\]

\[
= \frac{1}{-x} \sum_{k=1}^{n} \binom{n}{k} (-x)^k \quad \text{if } x \neq 0
\]

\[
= \frac{1}{-x} \left( -1 + \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} (-x)^k \right)
\]

\[
= \frac{1}{-x} (-1 + (1-x)^n). \quad \text{Then}
\]

\[
\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \int_{0}^{1} f_n
\]

\[
= \int_{0}^{1} \frac{1-(1-x)^n}{x} \, dx.
\]

With \(H_n\) equal to this integral we next show

\[
H_n = n \frac{1}{1}.
\]
This clearly holds if \( n=1 \); if \( 2 \leq n \) and

\[
H_n - 1 = \sum_{k=1}^{n-1} \frac{1}{k}, \text{ then }
\]

\[
H_n = \int_0^1 \frac{1-(1-x)(1-x)^{n-1}}{x} \, dx
\]

\[
= \int_0^1 \frac{1-(1-x)^{n-1}}{x} \, dx + \int_0^1 (1-x)^{n-1} \, dx
\]

\[
= H_{n-1} + \frac{1}{n},
\]

so the result follows by induction on \( n \).

**Solution by Bojan Basić**

We prove

\[
x \in \mathbb{R} \Rightarrow -\sum_{k=1}^{n} \frac{x^k}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{(x+1)^k}{k}.
\]

To show these polynomial functions are equal it is sufficient to prove that they are equal at a point and that their first derivatives are equal. They are obviously equal at \( x=0 \) (each side evaluates to 0); next

\[
\left( -n \frac{x^k}{k} \binom{n}{k} \right) = -n x^{k-1} \binom{n}{k}
\]
\[
\left( \sum_{k=1}^{n} \frac{1}{k} \right) \sum_{k=1}^{n} \binom{n}{k} \]

\[
= -\sum_{k=0}^{n} x^k \binom{n}{k} - 1
\]

\[
= -\frac{(x+1)^n - 1}{x}, \quad \text{and}
\]

\[
\left( \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{(x+1)^k}{k} \right)'
\]

\[
= -\sum_{k=1}^{n} (x+1)^{k-1}
\]

\[
= -\frac{(x+1)^n - 1}{(x+1)-1}
\]

\[
= -\frac{(x+1)^n - 1}{x},
\]

so the derivatives are equal. The assertion of the problem is the special case \(x = -1\).

Also solved by Bojan Basić (Serbia), Massoud Malek, and John Sayer