Problem for 2008 December

Proposed by Dan Jurca

Show that the $n$-th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

is odd if and only if $n$ is 1 less than a power of 2.

Solution 1 by the proposer

We shall use the following notation. For each positive integer $n$ and each prime number $p$ we let $\nu_p(n)$ equal the number of occurrences of $p$ in the factorization of $n$ as a product of primes; i.e.,

$$n = 2^{\nu_2(n)} \times 3^{\nu_3(n)} \times 5^{\nu_5(n)} \times \ldots$$

It is obvious that for each prime $p$ we have $\nu_p(mn) = \nu_p(m) + \nu_p(n)$; and we recall that for each positive integer $n$ and each prime $p$

$$\nu_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots$$

First we consider the case of positive and even $n$. By a straightforward induction we find that

$$\binom{2n}{n} = \frac{2^{n/2}}{(n/2)!} \times ((n+1) \times (n+3) \times (n+5) \times \ldots \times (2n-1))$$

so that

$$\nu_2\binom{2n}{n} = \frac{n}{2} - \nu_2((n/2)!)$$
Now we observe that the sum in the parentheses, being a finite sum, is strictly less than the infinite sum \( \frac{n}{4} + \frac{n}{8} + \frac{n}{16} + \ldots = \frac{n}{2} \); it follows that

\[
0 < \nu_2 \left( \binom{2n}{n} \right),
\]

and since \( n+1 \) is odd, it follows that if \( n \) is positive and even, then \( 0 < \nu_2(C_n) \), so that \( C_n \) is even.

Next suppose \( n \) is positive and odd. By a similar induction we find

\[
\binom{2n}{n} = \frac{2^{(n+1)/2}}{(n+1)/(n-1)/2)!} \times ((n+2)(n+4)(n+6)\ldots(x(2n-1)) \quad \text{so that}
\]

\[
C_n = \frac{1}{n+1} \binom{2n}{n}
\]

\[
= \frac{2^{(n+1)/2}}{(n+1)/(n-1)/2)!} \times ((n+2)(n+4)(n+6)\ldots(x(2n-1))
\]

\[
2^{(n-1)/2}
\]

\[
= \frac{2^{(n-1)/2}}{(n+1)/2)/(n-1)/2)!} \times ((n+2)(n+4)(n+6)\ldots(x(2n-1)) \quad \text{whence}
\]

\[
v_2(C_n) = -(n-1)/2 - v_2(((n+1)/2))
\]

\[
= \frac{n-1}{2} - \left( \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+1}{8} \right\rfloor + \left\lfloor \frac{n+1}{16} \right\rfloor + \ldots \right).
\]

We have \( C_2^{0} = C_0 = 1 = C_1 = C_2^{1} \) so suppose that \( n = 2^i - 1 \) for some \( i, 2 \leq i \). Then the sum in the parentheses equals \( 2^{i-2} + 2^{i-3} + \ldots + 1 = 2^{i-1} - 1 = (n-1)/2 \), so that \( v_2(C_n) = 0 \), and \( C_n \) is odd.
Otherwise, if \( n \) is odd but not of the form \( 2^i - 1 \), then there exist (unique) positive integers \( p \) and \( q \) such that \( n = 2^p + q \) and \( 0 < q < 2^p - 1 \). Hence the sum in the parentheses equals

\[
(2^{p-2} + \lfloor (q+1)/4 \rfloor) + (2^{p-3} + \lfloor (q+1)/8 \rfloor) + \ldots + (1 + \lfloor (q+1)/2^p \rfloor)
\]

\[
= 2^{p-1} - 1 + \lfloor (q+1)/4 \rfloor + \lfloor (q+1)/8 \rfloor + \ldots
\]

\[
< 2^{p-1} - 1 + (q+1)/2
\]

\[
= (2^p + q)/2 - 1/2
\]

\[
= n/2 - 1/2
\]

\[
= (n-1)/2
\]

so that \( 0 < v_2(C_n) \), and \( C_n \) is even.

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**Solution 2 by the proposer**

It is well-known that \( C_n \) is the number of binary trees with \( n \) nodes. Now for each binary tree \( T \) let \( T' \) be the binary tree obtained from \( T \) by interchanging the left and right subtrees at each node; restricting this to binary trees with \( n \) nodes we have a bijection \( \phi_n \) from the set of binary trees with \( n \) nodes to itself, and we observe that \( \phi_n \circ \phi_n \) is the identity. Further we observe that \( \phi_n(T) = T \) if and only if \( T \) is a full binary tree with \( n \) nodes, and this is possible if and only if \( n = 2^i - 1 \) for some natural number \( i \). It follows at once that the number of binary trees with \( n \) nodes, \( C_n \), is odd if and only if \( n \) is 1 less than a power of 2.

Also solved by Bojan Basic (Serbia) and John M. Sayer