Does there exist an infinite-dimensional normed linear space \((X, \| \|)\) such that each subspace of \(X\) is closed?

Solution by the proposer

Proposition. If \((X, \| \|)\) is an infinite dimensional normed space, then \(X\) includes a subspace which is not closed.

Proof.

Suppose \(\mathcal{B}\) is a (Hamel) basis for \(X\). Since by hypothesis \(\mathcal{B}\) is infinite there exists a denumerable subset \(\{\mathbf{b}_0, \mathbf{b}_1, \ldots\}\) of \(\mathcal{B}\) and since each \(\mathbf{b}_j \neq 0\), we can let \(x_0 = \mathbf{b}_0\) and \(1 \leq j \Rightarrow x_j = 1/(j(j+1)) \cdot \mathbf{b}_j/\|\mathbf{b}_j\|\). Hence \(1 \leq j \Rightarrow \|x_j\| = 1/(j(j+1))\).

For \(0 \leq n\) let \(Y_n = \{k_0x_0 + k_1x_1 + \cdots + k_nx_n \in X \mid k_0 + k_1 + \cdots + k_n = 0\}\). Then each \(Y_n\) is an \(n\)-dimensional subspace of \(X\), and \(Y_0 \leq Y_1 \leq \cdots \leq Y_n\). Let \(Y = \bigcup\{Y_n \mid 0 \leq n\}\). Then \(Y\) is a subspace of \(X\), and \(Y\) is not closed, as follows.

For \(1 \leq n\) let \(y_n = x_0 - (1/n)(x_1 + \cdots + x_n)\). Then \(y_n \in Y_n \leq Y\), and \((y_n)_{n=1}^{\infty} \to x_0\), since

\[
1 \leq n \Rightarrow \|x_0 - y_n\| = \|(1/n)(x_1 + \cdots + x_n)\| \\
= (1/n)\|x_1 + \cdots + x_n\| \\
\leq (1/n)(\|x_1\| + \cdots + \|x_n\|) \\
= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{j(j+1)} \\
= \frac{1}{n} \left(1 - \frac{1}{n+1}\right) \\
< \frac{1}{n}.
\]

However, \(\forall n:\ x_0 \notin Y_n\), so \(x_0 \notin Y\); hence \(Y\) is not a closed subspace of \(X\).

Remark. The factors \(1/(j(j+1))\) above can of course be replaced with any \(a_j\) where \(\sum_{j=1}^{\infty} a_j\) is a convergent series of positive terms.

Solution by Bojan Bašić (Serbia)

The answer is no, that is: Every infinite-dimensional normed linear space \((X, \| \|)\) has a subspace which is not closed. In order to prove this, we are going to prove that the kernel of a linear functional \(f\) on a normed space always form a subspace, and even more, that if \(f\) is unbounded then the corresponding subspace is not closed, thereby providing the answer by giving a construction of an unbounded \(f\) on a given space.

It is almost trivial to see that \(\ker f\) form a subspace. Indeed, for any \(x, y \in \ker f\) and \(a, b \in \mathbb{R}\) it holds \(f(ax + by) = af(x) + bf(y) = 0 + 0 = 0\), and therefore \(ax + by \in \ker f\).

Assume now that \(f\) is unbounded. Choose a sequence \(x_n \in X\) such that \((\forall n \in \mathbb{N})(\|x_n\| \leq 1)\) and \(\lim_{n \to \infty} f(x_n) = \infty\) (what is possible by unboundedness of \(f\)). Further, choose \(x\) which is not in \(\ker f\), and notice that for each \(n \in \mathbb{N}\) it holds \(f\left(x - \frac{f(x)}{f(x_n)}x_n\right) = f(x) - \frac{f(x)}{f(x_n)}f(x_n) = 0\), therefore \(x - \frac{f(x)}{f(x_n)}x_n \in \ker f\).

However, since it is easy to see that \(\lim_{n \to \infty} \left(x - \frac{f(x)}{f(x_n)}x_n\right) = x\), we have just found a sequence from \(\ker f\) which converges to an element not in \(\ker f\), what means that \(\ker f\) is not closed.

Finally, let us show how to construct an unbounded \(f\). Choose a sequence \(y_n\) of linearly independent elements of \(X\), and let \(f(y_n) = n\|y_n\|\). By the axiom of choice, sequence \(y_n\) can be extended to a vector space basis of \(X\), and we could define \(f\) in the other elements of this basis to be equal to 0. This definition on the basis uniquely determines \(f\) on each \(y \in X\), and this \(f\) is obviously unbounded.