Determine all integers \( n \) such that \( n \) equals the sum of two or more consecutive positive integers.

Solution by Dan Jurca

Proposition. The positive integer \( n \) equals the sum of two or more consecutive positive integers if and only if \( n \) does not equal a power of 2.

Proof.

First, if \( 0 \leq k, \ 2 \leq m, \) and \( n = (k + 1) + (k + 2) + \cdots + (k + m) \), then \( n = mk + m(m + 1)/2 \), so that \( 2n = 2mk + m(m + 1) = m(2k + m + 1) \). If \( m \) equals an even integer, then \( 2k + m + 1 \) equals an odd integer; in any case there exists an odd integer greater than 1 which divides \( 2n \), hence which divides \( n \); it follows that \( n \) does not equal a power of 2.

Next, suppose \( n \) does not equal a power of 2; then there exist a unique nonnegative integer \( a \) and a unique positive integer \( b \) such that \( n = 2^a(2b + 1) \). If \( 2^n \leq b \), let \( k = b - 2^a \); then \( 0 \leq k \) and

\[
(k + 1) + (k + 2) + \cdots + (k + 2^{a+1}) = 2^{a+1}k + (1 + 2 + \cdots + 2^{a+1})
\]

\[
= 2^{a+1}k + \frac{2^{a+1}(2^{a+1} + 1)}{2}
\]

\[
= 2^{a+1}k + 2^a(2^{a+1} + 1)
\]

\[
= 2^a(2k + 2^{a+1} + 1)
\]

\[
= 2^a(2b - 2^a + 2^{a+1} + 1)
\]

\[
= 2^a(2b + 1)
\]

\[
= n,
\]

so that \( n \) equals the sum of \( 2^{a+1} \) consecutive positive integers. If \( b < 2^n \), let \( k = 2^n - b - 1 \); then \( 0 \leq k \) and

\[
(k + 1) + (k + 2) + \cdots + [k + (2b + 1)] = (2b + 1)k + [1 + 2 + \cdots + (2b + 1)]
\]

\[
= (2b + 1)k + \frac{(2b + 1)(2b + 2)}{2}
\]

\[
= (2b + 1)k + (2b + 1)(b + 1)
\]

\[
= (2b + 1)[k + (b + 1)]
\]

\[
= (2b + 1)[(2^n - b - 1) + b + 1]
\]

\[
= (2b + 1) \cdot 2^n
\]

\[
= 2^n(2b + 1)
\]

\[
= n,
\]

so that \( n \) equals the sum of \( 2b + 1 \) consecutive positive integers.

Also solved by Bojan Bašić (Serbia), Massoud Malek, Bill Nico, and John M. Sayer