Problem for 2010 May

Communicated by Dan Jurca

a. Find all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that
\[ x \in \mathbb{R} \implies f(f(x)) = x. \]
b. Find a function \( f : \mathbb{R} \to \mathbb{R} \) such that
i. \( x \in \mathbb{R} \implies f(f(x)) = x \); and
ii. \( x \in \mathbb{R} \implies f(x) \neq x. \)

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Solution by Dan Jurca

a. Proposition. If \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( x \in \mathbb{R} \implies f(f(x)) = x \), then \( f = \text{id}_\mathbb{R} \).

Proof.

Since \( f \circ (f \circ f) = \text{id}_\mathbb{R} = (f \circ f) \circ f \), it follows that \( f \) is a homeomorphism (with inverse \( f \circ f \)); in particular, \( f \) is injective. Now suppose \( a \in \mathbb{R} \) and \( f(a) \neq a \); we derive a contradiction as follows.

Let \( b = f(a) \) and \( c = f(b) \). Since \( b = f(a) \neq a \), it follows that \( b \neq a \), and hence \( c \neq b \). Also \( c \neq a \), since otherwise \( f(c) = f(f(b)) = f(f(a)) = a \), so \( f(a) = a \), a contradiction. Thus \( a \neq b \), \( b \neq c \), and \( c \neq a \). We next consider the following two possibilities: \( f(a) < a \) or \( a < f(a) \). Recall that \( f(c) = f(f(b)) = f(f(a)) = a \).

If \( f(a) < a \), then \( b < a \). Then \( c < b < a \) or \( b < c < a \) or \( b < a < c \). In the first case we have \( f(b) < f(a) < f(c) \), so by the intermediate value theorem (and continuity of \( f \)) there exists \( \xi \in (c,b) \) such that \( f(\xi) = f(a) \). But then \( \xi \neq c \), contradicting injectivity of \( f \). In the second case we have \( f(a) < f(b) < f(c) \), so again there exists \( \xi \in (c,a) \) such that \( f(\xi) = f(b) \), another contradiction. In the third case we have \( f(a) < f(c) < f(b) \), so once again there exists \( \xi \in (b,a) \) such that \( f(\xi) = f(c) \), yet another contradiction. Therefore \( a \leq f(a) \).

The argument in case \( a < f(a) \) is similar. Since there does not exist \( a \in \mathbb{R} \) such that \( f(a) \neq a \), it follows that \( x \in \mathbb{R} \implies f(x) = x \), so \( f = \text{id}_\mathbb{R} \).

b. Recall the notation: \( \{ \} : \mathbb{R} \to \mathbb{R} \) by \( x \mapsto \{x\} = x - \lfloor x \rfloor \); i.e., \( \{x\} \) is the fractional part of \( x \). Then consider \( f : \mathbb{R} \to \mathbb{R} \) by \( x \in \mathbb{R} \implies f(x) = \lfloor x \rfloor + \{x \} + 1/3 \). Suppose \( x \in \mathbb{R} \), and say \( x = n + \theta \), where \( n \) is an integer and \( 0 \leq \theta < 1 \), so \( \lfloor x \rfloor = n \) and \( \{x\} = \theta \). (\( n \) and \( \theta \) are clearly uniquely determined.) Then \( f(x) = n + \{\theta + 1/3\} \), \( f(f(x)) = n + \{\theta + 2/3\} \), and \( f(f(f(x))) = n + \{\theta + 1\} = n + \theta = x \), so \( f(f(f(x))) = x \). Since \( \forall \theta : \{\theta + 1/3\} \neq \{\theta\} \), there exists no fixed point of \( f \).