Problem for 2010 July
Proposed by Dan Jurca

Suppose $1 \leq k \leq n \Rightarrow x_k \in \mathbb{R}$,
\[
1 \leq i \leq n, \ 1 \leq j \leq n \Rightarrow a_{ij} = \begin{cases} 
  x_i x_j & \text{if } i \neq j \\
  x_i x_j + 1 = x_i^2 + 1 & \text{if } i = j,
\end{cases}
\]
and consider the $n \times n$ matrix $A = (a_{ij})$.

a. Find the determinant of $A$.
b. Find the characteristic equation and the spectrum of $A$.
c. Find $n$ linearly independent eigenvectors of $A$.
d. Find the inverse of $A$.

Solution by the proposer

If each $x_k = 0$, then $A = I_n$, the $n \times n$ identity matrix, so the problem is trivial. Therefore, suppose $x_m \neq 0$ for some $m$, $1 \leq m \leq n$. If $b_{ij} = x_i x_j$ and $B = (b_{ij}) = A - I_n$ then the spectrum of $A$, $\sigma(A)$, equals $\{1 + \lambda \mid \lambda \in \sigma(B)\}$. With $s = x_1^2 + x_2^2 + \cdots + x_n^2$ we will show that the characteristic equation of $B$ is $q(\lambda) = (\lambda - s)\lambda^{n-1}$. Consider the vectors $v^1, v^2, \ldots, v^n$ as follows.

\[
v^m = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad k \neq m \Rightarrow v^k = \begin{bmatrix} v_1^k \\ v_2^k \\ v_3^k \\ \vdots \\ v_n^k \end{bmatrix}
\]

where $v_i^k = \begin{cases} 
  -x_k & \text{if } i = m \\
  x_m & \text{if } i = k \\
  0 & \text{if } i \neq m \text{ and } i \neq k
\end{cases}$

We compute $Bv^m$ and each $Bv^k$, $k \neq m$.

\[
(Bv^m)_i = \sum_{j=1}^n b_{ij} x_j = \sum_{j=1}^n (x_i x_j)x_j = \sum_{j=1}^n x_i (x_j x_j) = x_i \sum_{j=1}^n x_j^2 = sx_i
\]

whence

\[
Bv^m = s v^m; \quad \text{and}
\]

\[
k \neq m \Rightarrow (Bv^k)_i = \sum_{j=1}^n b_{ij} v_j^k = b_{im} v_m^k + b_{ik} v_k^k = (x_i x_m) \cdot -x_k + (x_i x_k) \cdot x_m = 0
\]

whence

\[
Bv^k = 0 = 0v^k.
\]

It follows that $\sigma(B) = \{s; 0\}$, and associated eigenvectors are $v^m; v^1, \ldots, v^{m-1}, v^{m+1}, \ldots, v^n$, which are linearly independent. (It is obvious that $\{v^k \mid k \neq m\}$ is a linearly independent set. Since $k \neq m \Rightarrow Av^k = 0v^k$ and $Av^m = sv^m \neq 0v^m$, it follows that $v^m$ is not a linear combination of the other $v^k$.) Hence $\sigma(A) = \{s+1; 1\}$ and associated eigenvectors are $v^n; v^1, \ldots, v^{m-1}, v^{m+1}, \ldots, v^n; \det(A) = s+1$; and the characteristic polynomial of $A$ is $p(\lambda) = (\lambda - (s + 1)) (\lambda - 1)^{n-1}$. Since $\det(A) = s + 1 \neq 0$, $A$ is invertible.

Next let

\[
c_{ij} = \begin{cases} 
  s + 1 - x_i^2 & \text{if } i = j \\
  -x_i x_j & \text{if } i \neq j,
\end{cases}
\]

and let $C = (c_{ij})$; we show that $AC = \det(A) \cdot I_n$. 

First we compute

\[(AC)_{ii} = \sum_{k=1}^{n} a_{ik} c_{ki}
\]

\[= \sum_{k=1}^{i-1} a_{ik} c_{ki} + a_{ii} c_{ii} + \sum_{k=i+1}^{n} a_{ik} c_{ki} \]

\[= \sum_{k=1}^{i-1} (x_i x_k) \cdot (-x_k x_k) + (x_i^2 + 1) \cdot (s + 1 - x_k^2) + \sum_{k=i+1}^{n} (x_i x_k) \cdot (-x_k x_k) \]

\[= -x_i^2 \sum_{k=1}^{i-1} x_k^2 + sx_i^2 + x_i^2 - x_i^4 + s + 1 - x_i^2 - x_i^2 \sum_{k=i+1}^{n} x_k^2 \]

\[= -x_i^2 s + sx_i^2 + s + 1 \]

\[= s + 1; \quad \text{and then} \]

\[i < j \Rightarrow (AC)_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj} \]

\[= \sum_{k=1}^{i-1} a_{ik} c_{kj} + a_{ii} c_{ij} + \sum_{k=i+1}^{j-1} a_{ik} c_{kj} + a_{ij} c_{jj} + \sum_{k=j+1}^{n} a_{ik} c_{kj} \]

\[= \sum_{k=1}^{i-1} (x_i x_k) \cdot (-x_k x_j) + (x_i^2 + 1) \cdot (-x_i x_j) + \sum_{k=i+1}^{j-1} (x_i x_k) \cdot (-x_k x_j) + \]

\[+ (x_i x_j) \cdot (s + 1 - x_j^2) + \sum_{k=j+1}^{n} (x_i x_k) \cdot (-x_k x_j) \]

\[= -x_i x_j \sum_{k=1}^{i-1} x_k^2 - x_i x_j \sum_{k=i+1}^{j-1} x_k^2 - x_i x_j \sum_{k=j+1}^{n} x_k^2 - x_i x_j \sum_{k=j+1}^{n} x_k^3 \]

\[= -x_i x_j \left[ s - x_i^2 - x_j^2 \right] - x_i^3 x_j + sx_i x_j - x_i x_j \]

\[= -x_i x_j \left[ s - x_i^2 - x_j^2 + x_i^2 - s + x_j^2 \right] \]

\[= 0; \]

and similarly \( j < i \Rightarrow (AC)_{ij} = 0 \). Therefore \( AC = (s + 1) \cdot I_n = \det A \cdot I_n \), so that, finally,

\[A^{-1} = \frac{1}{s + 1} C, \quad \text{where} \]

\[s = x_1^2 + \cdots + x_n^2. \]

Remark. If we let \( x = v^m = (x_1, x_2, \ldots, x_n)^T \), then we have \( A = xx^T + I_n \) and \( C = (s+2)I_n - A \). Hence, for example, \( Ax = (xx^T + I_n)x = x(x^T x) + x = xs + x = (s+1)x \), so that \( s+1 \in \sigma(A) \) with associated eigenvector \( x \). Next \( AC = A((s+2)I_n - A) = (s+2)A - A^2 = sA + I_n - I_n + 2A - A^2 = sA + I_n - (A - I_n)^2 = sA + I_n - xx^T xx^T = sA + I_n - xx^T x = sA + I_n - sxx^T = sA + I_n - s(A - I_n) = (s+1)I_n \), so that \( A^{-1} = 1/(s+1) \cdot C = 1/(s+1) \cdot ((s+2)I_n - A) \), as before.

Also solved by Matthew Felix and Massoud Malek. Massoud Malek has generalized this result.