Problem for 2011 February

Communicated by Dan Jurca

Show that of all \(n\)-gons inscribed in the unit circle the regular \(n\)-gon has greatest area.

Solution by Dan Jurca

If the central angles subtended by the sides of an inscribed \(n\)-gon are \(\theta_1, \theta_2, \ldots, \theta_n\), then the area of the polygon equals

\[
\frac{1}{2} \sum_{i=1}^{n} \sin \theta_i,
\]

and clearly \(1 \leq i \leq n \Rightarrow 0 < \theta_i < \pi\). Since the function \(f : [0, \pi] \rightarrow \mathbb{R}\) by \(f(\theta) = \sin \theta\) is concave it follows from Jensen’s inequality that

\[
\frac{\sum_{i=1}^{n} \sin \theta_i}{n} \leq \sin \frac{\sum_{i=1}^{n} \theta_i}{n} = \sin \frac{2\pi}{n}, \text{ so that }
\]

\[
\frac{1}{2} \sum_{i=1}^{n} \sin \theta_i \leq \frac{n}{2} \sin \frac{2\pi}{n},
\]

and this last quantity equals the area of a regular \(n\)-gon inscribed in the unit circle.

Also solved by Matthew Felix, Massoud Malek, Bill Nico, John M. Sayer, and Jan van Delden (the Netherlands)