Problem for 2011 August, September, and October

Proposed by Dan Jurca

For each integer \( n \), \( 3 \leq n \), let

- \( p_n \) = the perimeter of the regular \( n \)-gon inscribed in the unit circle,
- \( q_n \) = the perimeter of the regular \( n \)-gon circumscribed about the unit circle,
- \( A_n \) = the area of the regular \( n \)-gon inscribed in the unit circle, and
- \( B_n \) = the area of the regular \( n \)-gon circumscribed about the unit circle.

Show: if \( 3 \leq n \), then \( p_n \) more nearly approximates the perimeter of the circle than does \( q_n \); but if \( 4 \leq n \), then \( B_n \) more nearly approximates the area of the circle than does \( A_n \). That is, prove

\[
3 \leq n \Rightarrow |2\pi - p_n| < |2\pi - q_n| \quad \text{and} \quad 4 \leq n \Rightarrow |\pi - B_n| < |\pi - A_n|.
\]

Solution by the proposer

We find easily that

\[
3 \leq n \Rightarrow p_n = 2n \sin \frac{\pi}{n}, \quad q_n = 2n \tan \frac{\pi}{n}, \quad A_n = n \sin \frac{\pi}{n} \cos \frac{\pi}{n}, \quad B_n = n \tan \frac{\pi}{n},
\]
and that \( 3 \leq n \Rightarrow p_n < 2\pi < q_n \) and \( A_n < \pi < B_n \).

If \( f(\theta) = \sin \theta + \tan \theta - 2\theta \), then \( f(0) = 0 \), \( f'(\theta) = \cos \theta + \sec^2 \theta - 2 \), and \( f''(\theta) = -\sin \theta + 2 \sec^2 \theta \tan \theta = \sin(2\sec^2 \theta - 1) \). With \( \varphi(t) = 2\sec^3 t - 1 \) we have \( \varphi'(t) = 6\sec^2 t \tan t \), so \( 0 < t < \pi/2 \Rightarrow 0 < \varphi'(t) \).

Hence \( 0 < \theta < \pi/2 \Rightarrow 0 < f''(\theta) \), so \( f' \) increases in \([0, \pi/2]\). Therefore \( 0 < \theta < \pi/3 \Rightarrow 0 < f(\theta) \), so that \( 3 \leq n \Rightarrow 0 < \pi/n \leq \pi/3 \) and there follows that \( 0 < n \sin(n/\pi) + n \tan(n/\pi) - 2\pi/n \) so that \( 0 < n \sin(n/\pi) + n \tan(n/\pi) - 2\pi/n \), and thus \( 3 \leq n \Rightarrow 2\pi - 2n \sin(n/\pi) < 2n \tan(\pi/n) - 2\pi, \) so \( 2\pi - p_n < q_n - 2\pi \).

If \( g : (-\pi/2, \pi/2) \to \mathbb{R} \) by \( g(\theta) = 2\theta - \sin \theta \cos \theta - \tan \theta \), then \( g'(\theta) = 2 - \cos^2 \theta + \sin^2 \theta - \sec^2 \theta = 3 - 2\cos^2 \theta - \sec^2 \theta \). We observe that \( g'(0) = 0 \). Next, \( g''(\theta) = 4 \cos \theta \sin \theta - 2 \sec^2 \theta \tan \theta = 4 \sin \theta \cos \theta - 2 \sin \theta / \cos^3 \theta = 2 \sin \theta / (2 \cos \theta - \cos^3 \theta) \). If \( \psi : (-\pi/2, \pi/2) \to \mathbb{R} \) by \( \psi(\theta) = 2 \cos \theta - \cos^3 \theta \), then \( \psi'(\theta) = -2 \sin \theta + 3 \cos^2 \theta - \sin \theta = -(2 \sin \theta + 3 \sin \theta \cos^2 \theta) \). It follows that \( 0 < \theta < \pi/2 \Rightarrow \psi'(\theta) < 0 \), so that \( \psi \) decreases in \([0, \pi/2]\). Now \( \psi(0) = 1 \), and \( \psi(\pi/6) = 2 - 2\sqrt{3}/2 - 1/(\sqrt{3}/2)^3 = \sqrt{3} - 8/(3\sqrt{3}) = 9\sqrt{3}/9 - 8\sqrt{3}/9 = \sqrt{3}/3 > 0 \). Therefore \( 0 \leq \theta \leq \pi/6 \Rightarrow 0 < \psi(\theta) \). Since \( g''(\theta) = 0 \) it follows that \( 0 < \theta < \pi/6 \Rightarrow 0 < g''(\theta) \), so \( g' \) increases in \([0, \pi/6]\). Since \( g'(0) = 0 \), it follows that \( g \) increases in \([0, \pi/6]\). Hence \( 0 \leq \theta \leq \pi/6 \Rightarrow 0 < g(\theta) \). Thus \( 6 \leq n \Rightarrow 0 < 2 \cdot \pi/n - \sin \pi/n \cdot \cos \pi/n - \tan \pi/n \), so that \( \tan \pi/n - \pi/n < \pi/n - \sin \pi/n \cdot \cos \pi/n \), whence \( n \tan \pi/n - \pi < \pi - \pi/n \cdot \sin \pi/n \cdot \cos \pi/n \) from which \( 6 \leq n \Rightarrow B_n - \pi < \pi - A_n \). By straightforward computation \( B_4 - \pi < \pi - A_4 \) and \( B_5 - \pi < \pi - A_5 \).

Therefore, and finally, \( 4 \leq n \Rightarrow B_n - \pi < \pi - A_n \).