Problem for 2012 January

Communicated by Dan Jurca

Suppose $A$, $B$, and $C$ are three distinct points on an arbitrary parabola $P$. Prove that the area of the triangle $\Delta ABC$ is twice the area of the triangle bounded by the lines tangent to $P$ at $A$, $B$, and $C$.

Solution by Dan Jurca

Without loss of generality we may suppose the parabola is the curve in the $x$-$y$ plane defined by the equation $x^2 = 4py$, or $y = x^2/4p$. (This curve then consists of all points in the $x$-$y$ plane equidistant from the focus at $(0, p)$ and the line—the directrix—with equation $y = -p$.) Then suppose the three points are at $A: (a, a^2/4p)$, $B: (b, b^2/4p)$, and at $C: (c, c^2/4p)$. It follows that the area of the triangle with vertices at $A$, $B$, and $C$ is the absolute value of the following determinant.

\[
\begin{vmatrix}
  a & b & c \\
  a^2/4p & b^2/4p & c^2/4p \\
  1 & 1 & 1
\end{vmatrix}
\]

Equations of the lines tangent to the parabola at $A$ and $B$ are $y - a^2/4p = (a/2p)(x - a)$ and $y - b^2/4p = (b/2p)(x - b)$, respectively; and these line intersect at the following point.

\[
\left( \frac{a + b}{2}, \frac{ab}{4p} \right)
\]

The points of intersection of the other two pairs of tangent lines are similar; hence the area of the triangle bounded by the three tangent lines is the absolute value of the following determinant.

\[
\begin{vmatrix}
  a + b & b + c & c + a \\
  \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\
  ab & bc & ca \\
  \frac{4p}{4p} & \frac{4p}{4p} & \frac{4p}{4p} \\
  1 & 1 & 1
\end{vmatrix}
\]

By straightforward computation this determinant equals $-1/2$ times the determinant above.

Also solved by Lewis Felver and Kourosh Ghaderi