Problem for 2012 June

Proposed by Kourosh Ghaderi

Prove that if $G$ is a group with only finitely many subgroups, then $G$ is of finite order.

Solution by Dan Jurca

We consider two possibilities, as follows.

i. $g \in G \Rightarrow$ the order of $g$ is finite; or

ii. there exists $g \in G$ such that the order of $g$ is infinite.

In the first case, since $G = \bigcup \{\langle g \rangle \mid g \in G\}$, where $\langle g \rangle$ is the cyclic group generated by $g$, and there exist only finitely many such $\langle g \rangle$, it follows that (the underlying set) $G$ is the finite union of finite sets, so that $G$ is finite.

In the second case, suppose $g \in G$ and the order of $g$ is infinite. Then the cyclic group $\langle g \rangle$ generated by $g$ is isomorphic to the group $\mathbb{Z}$, the (abelian) group of integers under addition. (An isomorphism is defined by extending $g \mapsto 1$.) But there exist infinitely many subgroups of $\mathbb{Z}$. For if $m$ and $n$ are distinct positive integers, then the quotient groups $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ contain $m$ and $n$ elements, respectively. Therefore $G$ does not contain an element of infinite order.

Also solved by Massoud Malek and Winston Teitler