Problem for 2013 May

Proposed by Dan Jurca

Generalizing such formulas as \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \) and \( \sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta \), show that if \( s = \sin \theta \) and \( c = \cos \theta \), then for each positive integer \( n \)

\[
\sin n\theta = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} c^{n-(2k+1)} s^{2k+1}
\]

and

\[
\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} c^{n-2k} s^{2k}.
\]

Solution by the proposer

From Euler’s identity \( e^{i\theta} = \cos \theta + i \sin \theta \) we find the following.

\[
\cos n\theta + i \sin n\theta = e^{i(n\theta)}
\]

\[
= e^{n(i\theta)}
\]

\[
= (e^{i\theta})^n
\]

\[
= (\cos \theta + i \sin \theta)^n
\]

\[
= (c + is)^n
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (is)^k
\]

\[
= \sum_{k=0}^{n} i^k \binom{n}{k} c^{n-k} s^k
\]

and the result follows from the fact that \( i^2 = -1, \ i^3 = -i, \ i^4 = 1, \ldots \), and then by equating real and imaginary parts.

Also solved by Massoud Malek and Winston Teitler