Problem for 2015 January

Proposed by Dan Jurca

Determine all points \((x,y)\) in the Cartesian \(x-y\) plane such that
\[x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4.\]

Solution by the proposer

\[
x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4 \implies \\
x^3y + 4xy^3 + x^2 - 4xy + 4y^2 - 4 = 0 \implies \\
x^3y + 4xy^3 - 4xy + x^2 + 4y^2 - 4 = 0 \implies \\
(xy + 1)(x^2 + 4y^2 - 4) = 0 \implies \\
(xy + 1) \left(\frac{x^2}{4} + \frac{y^2}{1} - 1\right) = 0.
\]

Therefore if \(x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4\), then either \(xy + 1 = 0\) or \(x^2/4 + y^2/1 - 1 = 0\); so the points \((x,y)\) such that the given equation holds lie on either the hyperbola with equation \(y = -1/x\) or the ellipse with equation \(x^2/4 + y^2/1 = 1\). Conversely, if a point with coordinates \((x,y)\) lies on either the hyperbola with equation \(y = -1/x\) or on the ellipse with equation \(x^2/4 + y^2/1 = 1\), then \(xy + 1 = 0\) or \(x^2/4 + y^2/1 - 1 = 0\), so \(x^3y + 4xy^3 + x^2 - 4xy + 4y^2 = 4\). The set of points in question is therefore the union of these two curves.

Also solved by John M. Sayer and Winston Teitler

There were two incomplete solutions.