Problem for 2015 November
Proposed by Dan Jurca

We show that the series converges to $4 - \ln 16 = 1.227411\ldots$.

First, for each positive integer $n$ let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$, and recall that

$$\lim_{n \to \infty} (H_n - \ln n)$$

exists and equals a certain number, the Euler-Mascheroni constant, $\gamma \approx 0.57721$. We will need the following Lemma. $\lim_{n \to \infty} (H_{2n} - H_n) = \ln 2$.

Proof. Let $\varepsilon > 0$. Pick $n_1$ and $n_2$ such that

$$|H_{2n_1} - \ln 2n_1 - \gamma| < \varepsilon/2$$

and

$$|H_{n_2} - \ln n_2 - \gamma| < \varepsilon/2.$$ 

Hence

$$|H_{2n_1} - H_{n_2}| < \varepsilon.$$ 

Now

$$H_{2n_1} < H_{2n_1} - H_{n_2} < H_{2n_1} - H_{n_2} + H_{n_2} - H_{n_2} = H_{2n_1} - H_{n_2} + 1,$$

and

$$H_{2n_1} - H_{n_2} < H_{2n_1} - H_{n_2} + 1.$$ 

so

$$H_{n_2} < H_{2n_1} - 1.$$ 

Hence

$$H_{2n_1} - H_{n_2} < H_{2n_1} - H_{n_2} + 1.$$ 

and the proposition follows by induction on $n$.

Hence by the proposition and the lemma

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1/2)} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k(k+1/2)} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} [(4 - 4(H_{2n} - H_n)) - 4/(2n+1)] = 4 - 4\ln 2 = 4 - \ln 16.$$

Solution by Benjamin Thomas

$$\frac{1}{k(k+1/2)} = \frac{1}{2} \left[1/k - 1/(k+1/2)\right]$$

$$= 4 \left[1/(2k) - 1/(2k+1)\right],$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1/2)} = 4 \left[(1/2 - 1/3) + (1/4 - 1/5) + (1/6 - 1/7) + \cdots\right]$$

$$= 4 - 4 \left[1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots\right]$$

$$= 4 - 4\ln 2.$$

Also solved by Jan van Delden (the Netherlands), Massoud Malek, John M. Sayer, and Winston Teitler.

John M. Sayer found the following formula on the internet.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)} = \frac{1}{\alpha} [\psi(\alpha + 1) + \gamma]$$

where $\psi$ is the psi (or digamma) function.