The following is well known; the proposer learned of it only recently.

Prove that if $A$ is a square real or complex matrix, then $\det(e^A) = e^{\text{trace}(A)}$.

Solution by Dan Jurca

Suppose $A$ is $n \times n$. If $\lambda \in \text{spec}(A)$, (i.e., $\lambda$ is an eigenvalue of $A$), then for some nonzero $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$, $Ax = \lambda x$. Hence

$$e^A x = \left( I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \right) x$$
$$= x + Ax + \frac{A^2 x}{2!} + \frac{A^3 x}{3!} + \cdots$$
$$= 1 \cdot x + \lambda \cdot x + \frac{\lambda^2}{2!} \cdot x + \frac{\lambda^3}{3!} \cdot x + \cdots$$
$$= \left( 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right) x$$
$$= e^\lambda x.$$

So if the characteristic polynomial $p(x) = \det(xI_n - A)$ splits over $\mathbb{C}$ as $(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$, then for each of the (not necessarily distinct) eigenvalues $\lambda_j$, $j = 1, \ldots, n$ of $A$, $e^{\lambda_j}$ is an eigenvalue of $e^A$. It follows that $\text{spec}(e^A) = \{ e^\lambda \mid \lambda \in \text{spec}(A) \}$. (This is non-trivial: see Theorem 6 on page 312 of *The Theory of Matrices* by Peter Lancaster and Miron Tismenetsky.) Since the determinant of a matrix equals the product of the eigenvalues of the matrix,

$$\det(e^A) = e^{\lambda_1} e^{\lambda_2} e^{\lambda_3} \cdots e^{\lambda_n}$$
$$= e^{\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n}$$
$$= e^{\text{trace}(A)},$$

since the trace of a matrix equals the sum of the eigenvalues of the matrix.

Also solved by Massoud Malek and Benjamin Thomas