Problem for 2016 August
Proposed by Benjamin Thomas

Find the general solution in terms of elementary functions:

\[ 2x \left(1 + x\right) \frac{d^2y}{dx^2} + 3 \left(1 + x\right) \frac{dy}{dx} - y = 0 \]

Solution by the proposer

Any analytic solution can be expressed as a power series of the form \( \sum_{n=0}^{\infty} a_n x^n \).

Let

\[ y = \sum_{n=0}^{\infty} a_n x^n \]
\[ \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1} \]
\[ \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n (n - 1) a_n x^{n-2} \]

We notice the immediate consequence: \( \sum_{n=0}^{\infty} n a_n x^n = x \frac{dy}{dx} \)

Plugging these back into our differential equation and rearranging terms we arrive at \( \sum_{n=0}^{\infty} (2n - 1) (n + 1) a_n + (2n + 3) (n + 1) a_{n+1} x^n = 0 \).

In order to find the sequence, \( a_n \), we can set each coefficient of \( x^n \) equal to 0.

\[ (2n - 1) (n + 1) a_n + (2n + 3) (n + 1) a_{n+1} = 0 \]
\[ (2n - 1) a_n + (2n + 3) a_{n+1} = 0 \]

This allows us to arrive at

\[ \sum_{n=0}^{\infty} (2n - 1) a_n x^n + \sum_{n=0}^{\infty} (2n + 3) a_{n+1} x^n = 0 \]

and finally

\[ 2x \left(1 + x\right) \frac{dy}{dx} + (1 - x) y = a_0 \]

for some arbitrary constant, \( a_0 = y(0) \).

Putting this in standard form and using the integrating factor \( \mu = \frac{\sqrt{x}}{1 + x} \) we arrive at our solution.

\[ y = \frac{a_0}{2} \left( 1 + \frac{1 + x}{\sqrt{x}} \tan^{-1} \left( \sqrt{x} \right) \right) + C \frac{1 + x}{\sqrt{x}} \]
\[ y = C_1 \left( 1 + \frac{1 + x}{\sqrt{x}} \tan^{-1} \left( \sqrt{x} \right) \right) + C_2 \frac{1 + x}{\sqrt{x}} \]