Problem for 2016 October

Proposed by Matthew Hubbard

Suppose \( T \) is a triangle such that

- \( T \) is nondegenerate; \( i.e. \), the area of \( T \) is strictly positive; and
- the length of each side of \( T \) is an integer; and
- the perimeter of \( T \) is odd.

Prove that the area of \( T \) is irrational.

\[ \text{Solution by Dan Jurca} \]

Lemma. If \( n \) is an integer, then \( n^2 \) is congruent to 0, 1, or 4 (mod 8).

Proof.

If \( n \) is an integer, then (by the division theorem) there exist (unique) integers \( q \) and \( r \) such that \( n = 4q + r \) and \( 0 \leq r < 4 \). Hence \( n^2 = (4q + r)^2 = 16q^2 + 8qr + r^2 = 8(2q^2 + qr) + r^2 \), and since \( 0 \leq r < 4 \), it follows that \( r^2 \) is 0, 1, or 4, and the lemma follows.

Now suppose the lengths of the sides of \( T \) are \( a, b, \) and \( c \), each a positive integer. Since \( a + b + c \) is odd, it follows that either only one side is of odd length, or all three sides are of odd length.

So consider first for nonnegative integer \( x \) and positive integers \( y \) and \( z \), that \( a = 2x + 1 \), \( b = 2y \), and \( c = 2z \). Using Heron’s formula for the area \( A \) of \( T \), \( A = \sqrt{s(s-a)(s-b)(s-c)} \) where \( s = (a+b+c)/2 \), we compute as follows.

\[
\begin{align*}
  s &= (1 + 2x + 2y + 2z)/2; \\
  A &= \sqrt{s(s-a)(s-b)(s-c)} \\
  &= \sqrt{(1+2x+2y+2z)(-1-2x+2y+2z)(1+2x-2y+2z)(1+2x+2y-2z)/4}
\end{align*}
\]

It is well known and easy to prove that the square root of an integer \( N \) is rational if and only if \( N \) is a perfect square. Therefore \( A \) is rational if and only if the quantity, say \( Q_1 \), in the radical sign above is a perfect square. However, we find by a tedious computation (or using Mathematica) that

\[
(1 + 2x + 2y + 2z)(-1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z)
= -1 - 8x - 24x^2 - 32x^3 - 16x^4 + 8y^2 + 32xy^2 + 32x^2y^2 - 16y^4 + 8z^2 + 32xz^2 + 32x^2z^2 + 32y^2z^2 - 16z^4
= 7 + 8(-1 - x - 3x^2 - 4x^3 - 2x^4 + y^2 + 4xy^2 + 4x^2y^2 - 2y^4 + z^2 + 4x^2z^2 + 4x^2y^2 - 2z^4),
\]

which is not congruent (mod 8) to 0, 1, or 4. Therefore \( Q_1 \) is not a perfect square, and \( A \) is irrational.

Next, suppose for nonnegative integers \( x, y, \) and \( z, \) that \( a = 2x + 1, b = 2y + 1, \) and \( c = 2z + 1 \). Using Heron’s formula, we compute as follows.

\[
\begin{align*}
  s &= (3 + 2x + 2y + 2z)/2; \\
  A &= \sqrt{s(s-a)(s-b)(s-c)} \\
  &= \sqrt{(3+2x+2y+2z)(1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z)/4} \\
  &= \sqrt{Q_2}/4
\end{align*}
\]

Again, \( Q_2 \), the product of four integers, is an integer, and is \( A \) rational if and only if \( Q_2 \) is a perfect square. However, again tediously (or using Mathematica), we find

\[
Q_2 = (3 + 2x + 2y + 2z)(1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z)
= 3 + 8x - 8x^2 - 32x^3 - 16x^4 + 8y + 32xy + 32x^2y - 8y^2 + 32xy^2 + 32x^2y^2 - 32y^3 - 16y^4 + 8z^2 + 32xz^2 + 32x^2z^2 + 32yz^2 + 32y^2z^2 - 32z^3 - 16z^4
= 3 + 8(x - x^2 - 4x^3 - 2x^4 + 4xy + 4x^2y - y^2 + 4xy^2 + 4x^2y^2 - 2y^4 + z^2 + 4x^2z^2 + 4x^2y^2 + 4yz^2 + 4y^2z^2 - 4z^3 - 2x^4),
\]

which is not congruent (mod 8) to 0, 1, or 4. Therefore \( Q_2 \) is not a perfect square, and \( A \) is irrational.