Problem for 2017 February

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a. Suppose $p$ is a prime number, $n$ is a positive integer, $x$ is a positive integer, and

$$x \equiv a \pmod{p^n}$$

where $0 < a < p^n$. Show that there exists a positive integer $y$ such that

$$xy \equiv a \pmod{p^{n+1}} \; \text{and} \; \gcd(p, y) = 1.$$

b. Suppose $a$ and $b$ are integers, $2 \leq a$, $2 \leq b$, and $\gcd(a, b) = 1$. Show that there exists a positive integer $x$ such that

$$x \equiv a \pmod{ab},$$

but there does not exist an integer $y$ such that

$$xy \equiv a \pmod{a^2b}.$$