Problem for 2017 June

Proposed by Dan Jurca

1. Without evaluating either, determine which is greater: $\pi \sqrt{10}$ or $\sqrt{10} \pi$.

2. Show that $e$, the base of natural logarithms, is the only real number $a$ such that $0 < x \Rightarrow x^a \leq a^x$.

Solution by the proposer

The function $\varphi: (0, \infty) \to \mathbb{R}$ by $\varphi(x) = \ln x/x$ is differentiable, and since

$$0 < x \Rightarrow \varphi'(x) = \frac{1 - \ln x}{x^2}$$

it follows that $0 < x < e \Rightarrow 0 < \varphi'(x)$, so that $\varphi$ strictly increases in $(0, e]$; $e < x \Rightarrow \varphi'(x) < 0$, so $\varphi$ strictly decreases in $[e, \infty)$. Therefore $\varphi_{\text{max}} = \varphi(e) = 1/e$, and this maximum value is attained at no other point.

1. Since $e \leq a < b \Rightarrow \varphi(b) < \varphi(a)$, and

$\pi < \frac{22}{7}$, it follows that

$\pi^2 < \frac{484}{49}$

$< \frac{490}{49}$

$= 10$, so that

$\pi < \sqrt{10}$,

and since $e < 3 < \pi < \sqrt{10}$, we have $\varphi(\sqrt{10}) < \varphi(\pi)$. Therefore

$$\ln \sqrt{10} < \frac{\ln \pi}{\pi}, \quad \text{so}$$

$$\pi \ln \sqrt{10} < \sqrt{10} \ln \pi, \quad \text{whence}$$

$$\sqrt{10}^\pi < \pi^{\sqrt{10}}.$$

2. First, since $\varphi_{\text{max}} = \varphi(e) = 1/e$, we have $0 < x \Rightarrow \varphi(x) \leq 1/e$; i.e., $\ln x/x \leq 1/e$, so that

$$e \ln x \leq x = x \ln e;$$

hence $0 < x \Rightarrow x^e \leq e^x$.

Now suppose $0 < a$, and $0 < x \Rightarrow x^a \leq a^x$. Then (since $0 < e$) by hypothesis $e^a \leq a^e$; but then by what we have just shown, we find $e^a \leq a^e \leq e^a$. Therefore $e^a = a^e$. But then $a = e \ln a$, whence $\ln a/a = 1/e$, so that $\varphi(a) = \varphi_{\text{max}}$, which is attained only at $e$; therefore $a = e$.

We may summarize the above as follows.

\begin{align*}
0 < a < x \leq e & \Rightarrow \varphi(a) < \varphi(x) \Rightarrow a^x < x^a \\
\quad e \leq x < a & \Rightarrow \varphi(a) < \varphi(x) \Rightarrow a^x < x^a; \\
0 < x & \Rightarrow \varphi(x) \leq \varphi(e) \Rightarrow x^e \leq e^x
\end{align*}

Also solved by John M. Sayer