1. Find a linear fractional transformation which maps the half plane \( \{a + bi \mid a > b\} \) onto the disk \( \{a + bi \mid a^2 + b^2 < 4\} \), and determine its inverse.

2. Find all solutions \( x + iy \) of the equation \( \sin z = 2i \).

3. Let \( f \) be a function analytic in an open set \( A \subset \mathbb{C} \). Prove that if \( \text{Re} \ f \) is constant in \( A \), then \( f \) is constant in \( A \).

4. Let \( f \) be an entire function and \( a, b \in \mathbb{C} \) with \(|a| < R\) and \(|b| < R\).
   a. Prove:
   \[
   f(a) - f(b) = \frac{a - b}{2\pi i} \int_{|z|=R} \frac{f(z) \, dz}{(z - a)(z - b)}
   \]
   b. Use part a. to prove Liouville’s theorem: each bounded entire function is constant.
1. Let \((a_n)_{n=1}^\infty\) be a sequence of nonzero real numbers such that \(\lim_{n \to \infty} a_n = L \neq 0\). Prove that 
\[
\lim_{n \to \infty} \frac{1}{a_n} = \frac{1}{L}.
\]
(You must use an “\(\epsilon-N\)” argument.)

2. Prove that each Cauchy sequence in a metric space is bounded.

3. Let \(f : \mathbb{R}^2 \to \mathbb{R}\) as follows.
\[
f(x, y) = \begin{cases} 
x, & y = 0 
y, & x = 0 
1, & \text{elsewhere}
\end{cases}
\]
(a) Find \(\frac{\partial f}{\partial x}(0, 0)\) and \(\frac{\partial f}{\partial y}(0, 0)\).
(b) Prove that \(f\) is not differentiable at \((0, 0)\) by showing that \(f\) is not continuous at \((0, 0)\).

4. Let \(f : \mathbb{R} \to \mathbb{R}\) as follows.
\[
f(x) = \begin{cases} 
4, & x \text{ is rational} 
-2, & x \text{ is irrational}
\end{cases}
\]
Prove or disprove that \(f\) is Riemann integrable on any interval \([a, b] \subset \mathbb{R}\).
1. Let \( B_l = \{ [a, b) \mid a, b \in \mathbb{R} \text{ and } a < b \} \) be a collection of subsets of \( \mathbb{R} \).
   (a) Prove that \( B_l \) is a basis for a topology \( R_l \) of \( \mathbb{R} \) that is strictly finer than the standard topology on \( \mathbb{R} \).
   (b) Determine whether the topological space \( R_l \) is connected. Justify your answer.

2. Let \( X \) and \( Y \) be topological spaces and let \( f, g : X \to Y \) be continuous maps.
   Prove: If \( Y \) is Hausdorff, then \( A = \{ x \in X \mid f(x) = g(x) \} \) is closed in \( X \).

3. (a) Let \( C \) and \( D \) be disjoint nonempty open subsets of a topological space \( X \) such that \( X = C \cup D \).
   Prove: If \( Y \) is a connected subspace of \( X \), then \( Y \subseteq C \) or \( Y \subseteq D \).
   (b) Let \( (A_n)_{n=1}^{\infty} \) be a sequence of connected subspaces of a topological space \( X \) such that \( A_n \cap A_{n+1} \neq \emptyset \ \forall n \geq 1 \). Use (a) to prove that \( \bigcup \{ A_n \mid 1 \leq n \} \) is connected.

4. Prove the “tube lemma”: Let \( X \) and \( Y \) be topological spaces and let \( x \) be a point in \( X \). If \( Y \) is compact and \( N \) is an open neighborhood of \( \{x\} \times Y \) in \( X \times Y \), then there exists an open neighborhood \( U \) of \( x \) in \( X \) such that \( U \times Y \subseteq N \).
1. Find the general solution of the following initial value problem.

\[ X' = \begin{bmatrix} 5 & -1 \\ 9 & -1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

2. Use power series to find the solution about the point \( x_0 = 0 \) of the following equation. Give at least three nonzero terms of the series solution. Also, determine the radius of convergence.

\[(x^2 - 1)y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1\]

3. Prove that each Cauchy sequence in a metric space is bounded.

4. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) as follows.

\[ f(x, y) = \begin{cases} x, & y = 0 \\ y, & x = 0 \\ 1, & \text{elsewhere} \end{cases} \]

(a) Find \( \frac{\partial f}{\partial x}(0, 0) \) and \( \frac{\partial f}{\partial y}(0, 0) \).

(b) Prove that \( f \) is not differentiable at \( (0, 0) \) by showing that \( f \) is not continuous at \( (0, 0) \).
1. (a) Prove that there exists exactly one solution in the interval \([0, \infty)\) of the equation \(xe^x = 3\).
(b) Find an approximation \(\beta\) of the exact solution \(\alpha\) such that \(|\alpha - \beta| < 10^{-6}\).
(c) Prove that your approximation \(\beta\) is in fact within \(10^{-6}\) of the exact \(\alpha\).

Note: For this problem you may not use any graphing or root finding capabilities of your calculator.

2. (a) Determine a value of \(n\) such that using the composite Simpson’s rule with \(n\) subintervals of \([1, 3]\) will give an approximation of
\[
\int_1^3 (\ln x + x^3 - 2x + 1) \, dx
\]
with an error not exceeding \(10^{-3}\). Recall that the error term for \(\int_a^b f\) is \(\frac{b - a}{180} |f^{(4)}(\xi)|\) where \(h = (b - a)/n\) and \(\xi \in (a, b)\).
(b) Using your value of \(n\) find an approximation of the integral above. The composite Simpson’s rule can be written as follows.
\[
\int_a^b f \approx \frac{h}{3} \left[ f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]
\]
Here \(h = (b - a)/n\), and \(x_i = a + ih\) for \(i = 0, 1, 2, \ldots, n\).

3. Let \(I\) be the \(n \times n\) identity matrix.
(a) Prove that \(\|I\| \geq 1\) for each matrix norm.
(b) Use the result of part (a) to prove that if \(A\) is a nonsingular \(n \times n\) matrix, then \(\text{cond}(A) \geq 1\).

4. Write an algorithm to perform a single step of Gauss-Seidel iteration on an \(n \times n\) pentadiagonal system \((n \geq 3)\) of the following form.
\[
\begin{pmatrix}
a_1 & d_2 & e_3 & 0 & \cdots & 0 \\
b_1 & a_2 & d_3 & e_4 & \cdots & 0 \\
c_1 & b_2 & a_3 & d_4 & e_5 & \cdots \\
0 & c_2 & b_3 & a_4 & d_5 & e_6 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & e_n \\
\vdots & \ddots & \ddots & \ddots & 0 & e_n \\
0 & \cdots & \cdots & \cdots & d_{n-1} & a_n \\
0 & e_{n-1} & b_{n-2} & a_{n-1} & d_n \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\vdots \\
\vdots \\
x_{n-1} \\
x_n \\
\end{pmatrix}
= 
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\vdots \\
\vdots \\
y_{n-1} \\
y_n \\
\end{pmatrix}
\]
1. Use the Primal Simplex method to solve the following minimization problem.

\[
\begin{align*}
\text{minimize} & \quad -4x_1 + 5x_2 - 6x_3 + 3 \\
\text{subject to} & \quad 3x_1 + 4x_2 + 6x_3 \leq 8 \\
& \quad -x_1 + 3x_2 + 2x_3 \geq 4 \\
& \quad 2x_1 + 2x_2 + x_3 \geq 4 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

2. Consider the following minimization problem.

\[
\begin{align*}
\text{minimize} & \quad 9x_1 + 8x_2 + 8x_3 \\
\text{subject to} & \quad 5x_1 + 3x_2 + 3x_3 \leq 9 \\
& \quad x_1 + 2x_2 + x_3 \leq 4 \\
& \quad 6x_1 + 4x_3 \geq 11 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

The beginning and final tableaux in the Simplex method are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
</tr>
</thead>
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<tr>
<td>(x_1)</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_5)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(a_1)</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3/2</td>
</tr>
<tr>
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<td>0</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(x_3)</td>
<td>0</td>
<td>-9</td>
<td>1</td>
<td>-3</td>
<td>0</td>
<td>-5/2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>26</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>13/2</td>
</tr>
</tbody>
</table>

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer the questions.

(a) What is the range on the coefficient of \(x_1\) such that the basis variables do not change?

(b) What is the range on the coefficient of \(x_2\) such that the basis variables do not change?

(c) What is the range on the right hand side, \(b_3 = 11\), of the third constraint such that the basis variables do not change?
3. Solve the following transportation problem and give the final cost. The supplies are listed along the left, and the demands are listed along the top.

<table>
<thead>
<tr>
<th>50</th>
<th>75</th>
<th>100</th>
<th>125</th>
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</thead>
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</tr>
<tr>
<td>75</td>
<td>30</td>
<td>19</td>
<td>30</td>
</tr>
</tbody>
</table>

4. Consider the following maximization problem.

\[
\text{maximize} \quad 4x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad 2x_1 - 2x_2 + x_3 \leq 10 \\
\quad \quad \quad 6x_1 + 3x_2 - x_3 \leq 16 \\
\quad \quad \quad 3/2 x_1 + 4x_2 + 3x_3 \leq 15 \\
\quad \quad \quad -2x_1 + 6x_2 + 2x_3 \leq 10 \\
\quad \quad \quad x_j \geq 0
\]

Using the Complementary Slackness Theorem prove or disprove the following statement.

\[
\left[ \frac{42}{13}, 0, \frac{44}{13} \right] \quad \text{is an optimal solution.}
\]
1. Suppose we have a game where, at each turn, a person takes either one step forward or one step back. \(X_i\) measures how many steps the person is ahead of the origin (that is, negative values indicate the person is behind the origin) at the \(i\)th step. At the first trial \(X_0 = 0\). The probability of moving forward (that is, \(X_i = X_{i-1} + 1\)), \(p_f\) are as follows.

\[
p_f = \begin{cases} 
0.25 & \text{if } X_i > 0 \\
0.5 & \text{if } X_i = 0 \\
0.6 & \text{if } X_i < 0
\end{cases}
\]

(a) Give the probability distribution for \(X_2\).
(b) Give the probability distribution for \(X_3\).
(c) What are the expected value and variance of \(X_3\)?
(d) Give the probability distribution for \(X_3\) if \(X_1 = 1\).
(e) What are the expected value and variance of \(X_3\) if \(X_1 = 1\)?

2. Nicol (1994) defined the following probability density function for continuous \(X\).

\[
f(x) = \begin{cases} 
x & 0 < x \leq 1 \\
c/x^3 & 1 < x < \infty \\
0 & \text{otherwise}
\end{cases}
\]

(a) Find \(c\) such that \(f(x)\) is a probability density function.
(b) Find \(E(X)\).
(c) Find the median of \(X\).
(d) Find \(P(0.5 \leq X \leq 1.5)\).

3. A total of four buses carrying 145 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 30, 25, and 50 students. One of the students is randomly selected. Let \(X\) denote the number of students that were on the bus carrying the selected student. One of the four bus drivers is also randomly selected. Let \(Y\) denote the number of students on his/her bus.

(a) Without calculating the numbers, which of the expected values, \(E(X)\) or \(E(Y)\), do you think is larger? Why?
(b) Compute \(E(X)\) and \(E(Y)\).
(c) Compute the variance of \(X\) and the variance of \(Y\), \(Var(X)\) and \(Var(Y)\).
4. Company A has just developed a diagnostic test for a certain disease. The disease afflicts 1% of the population. The sensitivity of the test is the probability of someone testing positive, given that he or she has the disease, \( P(+) \mid D \), and the specificity of the test is the probability that someone tests negative, given that he or she does not have the disease, \( P(\neg) \mid D^c \). Assume that the sensitivity and specificity are each 95%.

Company B, which is a rival of Company A, offers a competing test for the disease. Company B claims that their test is faster and less expensive to perform than the test from Company A, is less painful, and yet has a higher overall success rate, where the overall success rate is defined as the probability that a randomly selected person is diagnosed correctly.

(a) The test from Company B can be described and performed very simply: no matter who the patient is, diagnose that he or she does not have the disease. Check whether the claim of Company B about overall success rates is true.
   i. Compute \( P(D+) \mid + \) and \( P(D^c \mid -) \) for Company A.
   ii. Compute \( P(D+) \mid + \) and \( P(D^c \mid -) \) for Company B.
   iii. Compare.

(b) Explain why the test from Company A may still be useful.

(c) Company A wants to develop a new test such that the overall success rate is higher than that of Company B.
   i. If the sensitivity and specificity are equal, how high does the sensitivity have to be to reach this goal?
   ii. If they can get the sensitivity equal to 1, how high does the specificity have to be to achieve the goal?
   iii. If they can get the specificity equal to 1, how high does the sensitivity have to be to achieve the goal?