Since neither the Real Analysis exam nor the Topology exam was given 2013 Autumn no copy of either exam is included here.
1. Let $G$ be a group, $H$ a subgroup of $G$, and let $N$ be a normal subgroup of $G$.
   a. Prove that $H \cap N$ is normal in $H$ and $HN = \{hn \mid h \in H \text{ and } n \in N\}$ is a subgroup of $G$.
   b. Let $\phi : HN \to H/(H \cap N)$ be defined by $\phi(hn) = hH \cap N$ for all $hn \in HN$.
      Prove that $\phi$ is a well-defined group homomorphism with kernel $N$. Conclude that $HN/N$ and $H/(H \cap N)$ are isomorphic.

2. Let $R$ be a commutative ring. Prove each of the following.
   a. $N = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}$ is an ideal of $R$.
   b. If $R$ is an integral domain and $I$ and $J$ are nonzero ideals of $R$, then $I \cap J \neq \{0\}$.

3. Prove that the center of a division ring is a field.

4. Let $A$ and $B$ be $n \times n$ matrices with real entries.
   a. Prove that if $A$ and $B$ are symmetric matrices, then $AB$ is symmetric if and only if $A$ and $B$ commute.
   b. Prove that if $A$ is a skew-symmetric matrix, then each element on the main diagonal of $A$ is zero.
Complex Analysis

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and write $g(z) = f(1/z)$. Prove that if $g$ has a pole at $z = 0$, then $f$ is a polynomial.

2. Find a linear fractional transformation which maps the unit circle to the real axis, sends $1$ to $0$, and sends $0$ to $i$.

3. Find all possible values of

$$
\int_\Gamma \frac{dz}{z^3 - z^2}
$$

for $\Gamma$ a smooth closed curve in $\mathbb{C}$ not passing through $0$ or $1$.

4. Use Liouville’s theorem to prove that if $f$ is an entire function such that

$$
\lim_{z \to \infty} f(z) = \infty,
$$

then there exists at least one zero of $f$ in $\mathbb{C}$.

Hint. Consider $g(z) = 1/f(z)$. 
1. Solve the following initial value problem.

\[ y'' - 3y' - 4y = 5e^{4x} + 2e^{x}, \quad y(0) = -\frac{1}{3}, \quad y'(0) = \frac{5}{3} \]

2. Use a power series to find the solution of the following differential equation.

\[ 2x^2y'' - xy' + (1 - x)y = 0 \]

3. Prove that if \( f : [a, b] \to [a, b] \) is continuous, then there exists a fixed point of \( f \).

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined as follows.

\[ f(x) = \sum_{n=1}^{\infty} \frac{x \cdot 3^n \cos(n^2x^2)}{(n/2)^n} \]

Prove that for each positive real number \( a \), \( f \) is uniformly convergent on \([-a, a] \).
1. a. Prove that there exists a unique real solution, say $\alpha$, of the equation $\ln x = \frac{1}{\sqrt{x}}$.  
   b. Find an approximation $\beta$ of $\alpha$ such that $|\alpha - \beta| < 10^{-6}$.  
   c. Prove that your approximation $\beta$ is in fact within $10^{-6}$ of $\alpha$.  
   Note: For this problem you may not use any graphing or rootfinding capabilities of your calculator.

2. Consider the following difference formula.
   
   $$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \frac{1}{6} f'''(x_0) h^2 + O(h^4)$$

   Observe that $\frac{f(x_0 + h) - f(x_0 - h)}{2h}$ is a $O(h^2)$ approximation of $f'(x_0)$.  
   Use this approximation to generate a $O(h^4)$ approximation of $f'(x_0)$ which involves $f(x_0 \pm h)$ and $f(x_0 \pm 2h)$.

3. Let $L$ be a lower triangular $n \times n$ matrix with $l_{ii} = 1$, $i = 1, \ldots, n$; let $U$ be an upper triangular $n \times n$ matrix with $u_{ii} \neq 0$, $i = 1, \ldots, n$; and let $b$ be an $n$-vector.  
   a. Write an algorithm to solve the equation $LUx = b$ for the $n$-vector $x$.  
   b. Let $t_n$ be the total number of multiplications and divisions (combined) performed when the algorithm in part a. is executed; write a formula for $t_n$.

4. Let $A$ be the matrix and $b$ the column vector as follows.

   $$A = \begin{bmatrix} 6 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 6 \end{bmatrix} \quad b = \begin{bmatrix} 8.50 \\ -14.00 \\ 14.75 \end{bmatrix}$$

   a. Prove that $A$ is positive definite.  
   b. Use Gauss-Seidel iteration with $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to approximate the solution of the equation $Ax = b$. Stop the iteration at $x^{(i)}$ where $\|x^{(i)} - x^{(i-1)}\|_\infty < 0.01$. 
Linear Programming

1. Consider the following maximization problem.

\[
\begin{align*}
\text{maximize} & \quad ax_1 + 2x_2 + 8x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + 3x_3 \leq 30 \\
& \quad x_1 + 2x_2 + 4x_3 \leq 40 \\
& \quad x_j \geq 0
\end{align*}
\]

Using the Complementary Slackness Theorem determine the values of \(a\) in the objective function such that \(x^* = (20, 10, 0)\) is the optimal solution.

2. Solve the following transportation problem, where supplies are listed in the first column, demands are written in the first row, and the various traveling costs are given in the table.

\[
\begin{array}{ccccccc}
& 27 & 90 & 45 & 36 & 47 \\
45 & 5 & 7 & 11 & 15 & 16 \\
90 & 8 & 6 & 10 & 12 & 15 \\
110 & 10 & 9 & 9 & 10 & 16 \\
\end{array}
\]

3. Solve the following using the Simplex Method (Primal method, not Dual method).

\[
\begin{align*}
\text{minimize} & \quad x + 2y + 3z \\
\text{subject to} & \quad x + y + z \geq 500 \\
& \quad x + 2y + 3z \geq 700 \\
& \quad -y + 3z \leq 0 \\
& \quad x, y, z \geq 0
\end{align*}
\]
4. Consider the following maximization problem.

\[
\begin{align*}
& \text{maximize} \quad x_1 + 2x_2 + 3x_3 \\
& \text{subject to} \quad x_1 - x_2 + x_3 \geq 4 \\
& \quad x_1 + x_2 + 2x_3 \leq 8 \\
& \quad x_1 - x_3 \geq 2 \\
& \quad x_j \geq 0
\end{align*}
\]

The beginning and final tableaux are given in the table below.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$a_2$</td>
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<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer each question.

(a) Suppose a new variable, $x_7$, is added. This variable appears in the maximization function as $4x_7$, and in the first and third constraints, respectively, as $-10x_7$ and $5x_7$. What is the new solution?

(b) Suppose the following constraint is added. What is the new solution?

\[
x_1 - 2x_2 + \frac{1}{2}x_3 \geq 3
\]

(c) What is the range on $c_2$, the coefficient of $x_2$ in the maximization function, such that the basis variables do not change?
Probability

1. Some kinds of statistical tests, especially nonparametric ones, assume that the populations are continuous, so that there are no ties in the data. However, in practice continuous data must be rounded, and there is some chance that rounding will cause ties.

   (a) Suppose six observations are taken at random from the uniform distribution on the interval \([0,1)\), and then rounded down to the nearest tenth of a unit. Thus there are only ten possible rounded values, and they are equally likely. What is the probability that there will be at least one tie among the rounded values? Give a numerical answer.

   (b) Suppose twenty-five observations are taken from the uniform distribution on the interval \([0,1)\), and then rounded down to the nearest 1,000th of a unit. Give a combinatorial expression for the probability of no ties among the rounded data.

   (c) It has been shown that the number of ties in part (b) has approximately a Poisson distribution with mean \(C(25,2)/1,000 = 25(24)/2,000 = 0.3\). You do not need to prove this result, but use it to approximate the answer to part (b).

2. Suppose the probability density function of a random variable \(X\) is given by the following.

   \[
   f(x) = \begin{cases} 
   a + bx & \text{if } -1 \leq x \leq 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   (a) What conditions must \(a\) and \(b\) satisfy?

   (b) Find the cumulative distribution function (CDF) of this distribution.

   (c) Based on the results in (a) determine the values of \(a\) and \(b\) for which the expectation of \(X\) is maximized.
3. You may use only the following definitions for this problem.

\[ E(Z) \equiv \mu_z \]
\[ \sigma_z^2 \equiv Var(Z) = E(Z^2) - \mu_z^2 \geq 0 \]
\[ \sigma_{wz} \equiv Cov(W, Z) = E(WZ) - \mu_w \mu_z \]
\[ \rho_{wz} \equiv Corr(W, Z) = \frac{\sigma_{wz}}{\sigma_w \sigma_z} \]

(a) Find the variance of \((X - Y)\) in terms of \(\sigma_x, \sigma_y, \) and \(\sigma_{xy}.\)

(b) Statisticians sometimes use a technique called “paired differences” in an effort to reduce variability of an outcome. (For example, one might use this technique for before and after measurements with a treatment such as a weight loss program.) If \(\sigma_x = \sigma_y,\) what must the correlation between \(X \) and \(Y\) be so that the variance of the difference is less than the variance of the value itself? (Give a range of values, not a single value.)

4. For a diagnostic test for a specific disease sensitivity is defined as the probability of getting a positive response on the test given that the individual has the disease, and specificity is defined as the probability of getting a negative result given that the individual does not have the disease. A test for disease \(A\) has 95% sensitivity and 98% specificity. Suppose 3% of the population has disease \(A.\)

\textit{Use four decimal places in your answers.}

(a) What is the probability that the person has the disease \textit{and} tests positive?

(b) What is the probability that the person does not have the disease \textit{and} tests negative?

(c) What is the probability that a person selected at random from the population will test positive?

(d) Find the probability that a person who tested positive actually has the disease.

(e) Find the probability that a person who tested negative actually does not have the disease.