1. Let \((G, \circ)\) be a group and \(g \in G\). Define \(a \bullet b = a \circ g \circ b\) for all \(a\) and \(b\) in \(G\).
   Prove that \((G, \bullet)\) is a group.

2. Let \(R\) be a commutative ring with unity 1 and suppose \(I\) is an ideal of \(R\).
   a. Prove that \(I\) is a prime ideal of \(R\) if and only if \(R/I\) is an integral domain.
   b. Determine whether \(\mathbb{Z} \times 6\mathbb{Z}\) is a prime ideal of \(\mathbb{Z} \times \mathbb{Z}\) where \(\mathbb{Z}\) is the ring of integers.
      Justify your answer.

3. Let \(\mathbb{Q}\) be the field of rational numbers and \(\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}\).
   a. Prove that \(\mathbb{Q}(\sqrt{5})\) is a subfield of \(\mathbb{R}\), the field of real numbers.
   b. Find all possible nontrivial ring homomorphisms \(\phi : \mathbb{Q}(\sqrt{5}) \to \mathbb{Q}(\sqrt{5})\) such that \(\phi(a) = a\) for each \(a \in \mathbb{Q}\).

4. Let \(A\) be a real \(n \times n\) matrix with characteristic polynomial \(p(\lambda)\), and let \(m(\lambda)\) be the monic polynomial of least degree such that \(m(A) = O\). \([m(\lambda)\) is the minimal polynomial of \(A\), and \(O\) is the zero matrix.\]
   Prove that \(m(\lambda)\) divides \(p(\lambda)\), and \(m(\lambda) = 0\) for each eigenvalue \(\lambda\) of \(A\).
1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f(z + \pi) = f(z)$ and $f(z + i) = f(z)$ for all $z \in \mathbb{C}$. Prove that $f$ is a constant function.

2. a. State Cauchy’s inequality for derivatives of an analytic function.

   b. Given a function $f$ analytic in $D = \{z \in \mathbb{C} \mid |z| \leq 10\}$ with $f(1) = 0$ and $f''(0) = i$, prove that $|f(z_0)| = \sqrt{2013}$ for some $z_0 \in D$.

3. Find all Laurent expansions centered at 0 for
   \[ f(z) = \frac{1}{z^4(1 + z^3)}, \]
   and give the region of convergence for each expansion.

4. Find a polynomial $p(z) = az^2 + bz + c$ such that
   \[
   \int_C \frac{p(z) \, dz}{z + 1} = 4\pi i, \quad \int_C \frac{p(z) \, dz}{z^2} = 8\pi i, \quad \text{and} \quad \int_C \frac{p(z) \, dz}{(z - 1)^2} = 12\pi i
   \]
   where $C$ is the closed contour illustrated.
1. a. Complete the “\( \varepsilon-N \)” definition:
A sequence \( (a_n)_{n=0}^{\infty} \) of real numbers converges to a limit \( L \) iff . . . .
b. Use the definition to prove the following.
   If \( (x_n)_{n=0}^{\infty} \) and \( (y_n)_{n=0}^{\infty} \) are real sequences, if \( (x_n)_{n=0}^{\infty} \) is bounded and \( (y_n)_{n=0}^{\infty} \rightarrow 0 \), then \( (x_ny_n)_{n=0}^{\infty} \rightarrow 0 \).

2. For each positive integer \( n \) define the function \( f_n : [0, 1] \rightarrow \mathbb{R} \) by \( f_n(x) = nx(1-x^2)^n \).
a. Find \( f(x) = \lim_{n \rightarrow \infty} f_n(x) \) for \( x \in [0, 1] \).
b. Determine whether \( \lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx \).
c. Prove or disprove: \( (f_n)_{n=1}^{\infty} \) converges uniformly to \( f \) in \( [0, 1] \).

3. Prove Cauchy’s Generalized Mean Value Theorem: If \( f, g : [a, b] \rightarrow \mathbb{R} \) are continuous on \( [a, b] \) and differentiable in \( (a, b) \), then there exists \( \xi \in (a, b) \) such that
   \[
f'(\xi)[g(a) - g(b)] = g'(\xi)[f(a) - f(b)].
   \]

4. Find the function \( f \) the Maclaurin series of which is \( \sum_{n=1}^{\infty} n^4 x^n \).
1. Let the set \( \mathbb{R} \) of real numbers have the “particular point” topology \( \mathcal{T} \) given as follows. 
\[ \mathcal{T} = \{ \emptyset \} \cup \{ U \subset \mathbb{R} \mid 6 \in U \}. \]
   a. With \( A = (-1, 1) \subset \mathbb{R} \) find \( \text{int} A \), \( \overline{A} \), and \( \partial A \).
   b. Prove or disprove: \((\mathbb{R}, \mathcal{T})\) is Hausdorff.
   c. Prove or disprove: \((\mathbb{R}, \mathcal{T})\) is compact.

2. Let \( X \) and \( Y \) be topological spaces with \( Y \) compact. Prove that \( \pi : X \times Y \to X \) given by \( \pi(x, y) = x \) for each \((x, y) \in X \times Y\) is a closed map.

3. Let \( X \) be a connected topological space and \( f, g : X \to [0, 1] \) be continuous maps with \( f \) surjective. Prove that \( f(x) = g(x) \) for some \( x \in X \).

4. Let \( X \) be a \( T_1 \) topological space (so 1-point subspaces of \( X \) are closed in \( X \)). Prove that \( X \) is normal iff for each closed subset \( A \) and each open subset \( U \) with \( A \subset U \) there exists an open subset \( V \) with \( A \subset V \subset \overline{V} \subset U \).
1. Find two linearly independent series solutions about $x = 0$ of the following differential equation.

$$2x^2y'' + 3xy' - (x^2 + 1)y = 0$$

Show the first four nonzero terms of each solution.

2. Find the Fourier series for the following function on the interval $[-\pi, \pi)$.

$$f(x) = \begin{cases} 
2 \cos x & \text{if } x \in [-\pi, 0) \\
-\cos x & \text{if } x \in [0, \pi) 
\end{cases}$$

Sketch the graph of the function to which the series converges for three periods, $[-3\pi, 3\pi]$.

3. a. Complete the “$\varepsilon-N$” definition:

A sequence $(a_n)_{n=0}^\infty$ of real numbers converges to a limit $L$ iff . . . .

b. Use the definition to prove the following.

If $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ are real sequences, if $(x_n)_{n=0}^\infty$ is bounded and $(y_n)_{n=0}^\infty \to 0$, then $(x_n y_n)_{n=0}^\infty \to 0$.

4. For each positive integer $n$ define the function $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = nx(1-x^2)^n$.

a. Find $f(x) = \lim_{n \to \infty} f_n(x)$ for $x \in [0, 1]$.

b. Determine whether $\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx$.

c. Prove or disprove: $(f_n)_{n=1}^\infty$ converges uniformly to $f$ in $[0, 1]$. 
1. a. Prove that there exist exactly two real solutions of the equation \( x^2 = 1 + \sin x \).
   b. Let \( \alpha \) be the smaller solution of the equation; find an approximation \( \beta \) of \( \alpha \) such that \( |\alpha - \beta| < 10^{-6} \).
   c. Prove that your approximation \( \beta \) is in fact within \( 10^{-6} \) of \( \alpha \).
   Note: For this problem you may not use any graphing or rootfinding capabilities of your calculator.

2. Use Taylor’s theorem to show that the truncation error in the following approximation is \( O(h) \).
   \[
   f''(x_0) \approx \frac{f(x_0) - 2f(x_0 + h) + f(x_0 + 2h)}{h^2}
   \]
   Assume \( f''' \) is continuous.

3. Recall the following
   - **Definition.** A symmetric \( n \times n \) matrix \( A \) is **positive definite** if \( x^TAx > 0 \) for each nonzero \( x \in \mathbb{R}^n \).

   Let \( A \) be a symmetric positive definite matrix. Using the definition prove
   a. \( A \) is nonsingular; and
   b. each entry on the main diagonal of \( A \) is strictly greater than 0.

4. Consider the following matrix.
   \[
   A = \begin{bmatrix}
   7 & -2 & 0 \\
   1 & -5 & 2 \\
   0.3 & 0.7 & 9
   \end{bmatrix}
   \]
   a. Using the Gershgorin Circle Theorem what can you say about the locations of the eigenvalues of \( A \)?
   b. Give the smallest interval \([a, b]\) which, by the Gershgorin Circle Theorem, contains the spectral radius of \( A \).
   c. Starting with the vector \( x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) perform two iterations of the Power Method to approximate the dominant eigenvalue of \( A \) and an associated eigenvector.
1. Consider the following linear programming problem.

\[
\begin{align*}
\text{maximize} & \quad 2x_1 - x_2 + x_3 \\
\text{subject to} & \quad 3x_1 + x_2 + x_3 \leq 60 \\
& \quad x_1 - x_2 + 2x_3 \leq 10 \\
& \quad x_1 + x_2 - x_3 \leq 20 \\
& \quad x_j \geq 0
\end{align*}
\]

The solution is \((15, *, 0)\) where * is some nonzero number. Use the Complementary Slackness Theorem to determine the value of *, and show that \((15, *, 0)\) is optimal.

2. The media mix problem occurs when a company has several media options in which to place ads, and is trying to reach different groups of people with the ads. Suppose that Essential Electronics has products which appeal to children, and men and women. They want to place ads on a typical children’s show, “Sponge Bob”, and two adult shows, “Friends”, and “The Daily Show.” The numbers of children, men, and women who watch these shows, and the unit costs of airing ads, are given in the following table.

<table>
<thead>
<tr>
<th>Program</th>
<th>Number Watching (millions)</th>
<th>Sponge Bob</th>
<th>Friends</th>
<th>The Daily Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>Children</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Men</td>
<td>1</td>
<td>10</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Women</td>
<td>1</td>
<td>20</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

| Unit Cost ($) | 3,000 | 25,000 | 8,000 |

Essential Electronics wants to ensure that a desired number of people in each group see the ads, as given in the following table.

<table>
<thead>
<tr>
<th>Group</th>
<th>Needed Exposure (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Children</td>
<td>18</td>
</tr>
<tr>
<td>Men</td>
<td>104</td>
</tr>
<tr>
<td>Women</td>
<td>134</td>
</tr>
</tbody>
</table>

Pose a linear programming problem to determine how many ads should be placed on each show to minimize the total cost and ensure that the desired numbers of people are seeing the ads. Use the Primal Simplex method to solve the problem.
3. Consider the following linear programming problem.

\[
\begin{align*}
\text{minimize} & \quad 2x_1 + 3x_2 + 4x_3 + 3x_4 \\
\text{subject to} & \quad x_1 + x_2 \leq 4 \\
& \quad x_3 + x_4 \leq 5 \\
& \quad x_1 + x_3 \geq 3 \\
& \quad x_2 + x_4 \geq 6 \\
& \quad x_j \geq 0
\end{align*}
\]

The following table gives the initial and final tableaux.

<table>
<thead>
<tr>
<th></th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>(x_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_5)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_6)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(a_1)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(a_2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>(z)</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(x_6)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(x_1)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(x_4)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(z)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Please answer the following questions using Sensitivity Analysis, with each question referring back to the original problem.

a. Suppose the same amount, \(\lambda\), is added to the right hand sides, \(b_1\) and \(b_2\), in the first two constraints. What is the range over which \(\lambda\) can vary without causing a change in the set of basic variables?

b. Suppose a new constraint is added: \(x_1 + x_4 \leq 7\). Find the new solution.

c. Suppose a new variable is added with a coefficient of \(-2\) in the objective function, and the vector of coefficients \((1, -1, 1, -1)^T\) in the constraints. Does the solution change? If so, what is the new solution?
4. Consider the Primal and Dual problems.

\[
\begin{align*}
\text{Primal Problem} & \quad \text{maximize} & & c^T x \\
& \quad \text{subject to} & & Ax \leq b \\
& & & x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{Dual Problem} & \quad \text{minimize} & & b^T y \\
& \quad \text{subject to} & & A^T y \geq c \\
& & & y \geq 0
\end{align*}
\]

Prove the following lemma. Any results you use in your proof must also be proven.

**Lemma:** If the objective function of the primal problem is unbounded above on the feasible region, then the dual problem is infeasible.
1. A firm is considering using the Internet to supplement its traditional sales methods. From the data of similar firms it is estimated that one of every 1,000 Internet hits results in a sale. Suppose the firm has 2,500 hits in a single day.

   a. Write an expression for the probability that there are less than six sales; do not complete the calculation.

   b. What assumptions are needed to write the expression in part a.?

   c. Use a Poisson approximation to compute the probability that less than six sales are made.

   Recall that the p.d.f. of Poisson(c) is \( P(X = k) = e^{-c} \frac{c^k}{k!} \).

   d. Why is a Poisson approximation better than a Normal approximation in this case?

2. Let \( X \) be a random variable with probability density function

\[
f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise}. \end{cases}
\]

Let \( Y|X \) be uniform(0, X). That is,

\[
f(y|X) = \begin{cases} 1/X, & 0 < y < X \\ 0, & \text{otherwise}. \end{cases}
\]

   a. Find the joint distribution of \( X \) and \( Y \); i.e., \( f_{X,Y}(x,y) \).

   b. What is the expected value of \( X \)?

   c. What is the expected value of \( Y|X \)?

   d. What is the expected value of \( XY \)?

   e. What is the expected value of \( Y \)?
3. Two factory workers are making “widgets” in an assembly line. Let $X$ represent the proportion of time (during an eight hour day) that the first worker spends on making widgets, and let $Y$ represent the proportion of time (during an eight hour day) that the second worker spends on making widgets. The joint distribution of $X$ and $Y$ is

$$f(x, y) = \begin{cases} 2x & 0 < x \leq 1, \ 0 < y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The first factory worker produces 27 widgets if she works all eight hours, and the second factory worker produces 20 widgets if she works all eight hours. Let $W = 27X + 20Y$ = the total number of widgets produced in a day.

a. Find the expected number of widgets to be produced by each of the workers separately. (Hint: Find the marginal density function for $X$ and the marginal density function for $Y$.)

b. Find the expected number of widgets to be produced by both workers in a single day.

c. Find the variance for the number of widgets to be produced by both workers in a single day.

d. Recall that Chebyshev’s theorem says that for any random variable $U$ with finite mean $\mu$ and finite variance $\sigma^2$, we have the following inequality.

$$P(|U - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Use your answers to b. and c. above to give bounds on the number of widgets that will be produced for at least 75% of the days.
4. The policy holders insured by an insurance company are classified into three risk categories \((I, II, III)\). It is assumed that each policy holder has at most one claim each year.

<table>
<thead>
<tr>
<th>Type of Risk</th>
<th>Probability of Annual Claim</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.1</td>
</tr>
<tr>
<td>II</td>
<td>0.2</td>
</tr>
<tr>
<td>III</td>
<td>0.4</td>
</tr>
</tbody>
</table>

One randomly chosen policy holder has three claims during six years.

a. Compute the conditional probabilities of three claims in six years for each risk category; \(i.e.,\) compute \(P(3|I)\) and the other two similar probabilities.

b. The probabilities of each risk category are

\[
P(I) = 0.7, \quad P(II) = 0.2, \quad \text{and} \quad P(III) = 0.1.
\]

Compute the probability of three claims being made for all policy holders; \(i.e.,\) compute \(P(3)\).

c. Given that a policy holder has three claims in six years, compute the probability that the policy holder is a member of risk category \(I; \ i.e.,\) compute \(P(I|3)\).

d. Compute the other two probabilities \(P(II|3)\) and \(P(III|3)\). Check that your calculations are correct. Explain.

e. What is the probability of a claim by a policy holder in year 7?