1. Given group homomorphisms \( f : G \rightarrow H \) and \( g : G \rightarrow H \), and

\[
K = \{ a \in G \mid f(a) = g(a) \},
\]

prove or disprove: \( K \) is a subgroup of \( G \).

2. Let \( I \) be an ideal in a commutative ring \( R \) with multiplicative identity 1 such that \( I \neq R \) and

\[
\forall a, b \in R, \ ab \in I \Rightarrow a \in I \text{ or } b \in I.
\]

Prove that \( R/I \) is an integral domain.

3. Let field \( E \) be a finite extension of field \( F \). Prove that \( E \) is an algebraic extension of \( F \).

4. Let \( U \) and \( W \) be subspaces of a vector space \( V \) over a field \( F \). Prove each of the following.
   (a) If \( U \cap W = \{0\} \), then for all nonzero vectors \( u \in U \) and \( w \in W \) the set \( \{u, w\} \) is linearly independent.
   (b) If \( V \) is finite dimensional and \( \dim_F(V) < \dim_F(U) + \dim_F(W) \), then \( U \cap W \neq \{0\} \).
1. Let $D$ be an open connected subset of $\mathbb{C}$, and let $f : D \to \mathbb{C}$ be analytic in $D$. Prove that if $f$ is analytic in $D$, then $f$ is constant in $D$.

2. (a) Determine for each of the functions
\[
f(z) = \frac{1}{\sin z} \quad g(z) = \sin \left(\frac{1}{z}\right) \quad h(z) = \frac{\sin z}{z}\]
whether $z = 0$ is a removable singularity, a pole, or an essential singularity.

(b) Suppose $C$ is the unit circle oriented clockwise; use residues to compute this integral.
\[
\oint_C \sin \left(\frac{1}{z}\right) (z^2 + 1) \, dz
\]

3. Let $z_0$ and $z_1$ be complex numbers such that $|z_0| < |z_1| < R$ and
\[
\int_{C_R} \frac{f(z) \, dz}{(z - z_0)(z - z_1)} = 0
\]
for some entire function $f$ where $C_R$ is the circle of radius $R > 0$ centered at the origin and traversed once counterclockwise. Prove that $f(z_0) = f(z_1)$.

4. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound $u_0$; i.e., $\forall (x, y) : u(x, y) \leq u_0$. Prove that $u(x, y)$ is constant throughout the plane.
Hint: Apply Liouville’s theorem to $g(z) = e^{f(z)}$. 
1. a. Complete the “ε-N” definition: A sequence \((a_n)_{n=0}^{\infty}\) converges to a limit \(L\) if . . . .
   
b. Use the ε-N definition to prove directly that
   
   \[
   \text{if } a_n = \frac{2n^3 + 2}{3n^3 - 1}, \text{ then } (a_n)_{n=0}^{\infty} \text{ converges to } \frac{2}{3}.
   \]

2. Let the function \(f\) be differentiable at \(x = a\), and let \(p(x)\) be the tangent line to the graph of \(f\) at \(x = a\). Prove
   
   \[
   \lim_{x \to a} \frac{|f(x) - p(x)|}{|x - a|} = 0.
   \]

3. Suppose \((f_n)_{n=1}^{\infty}\) is a sequence of continuous functions \(f_n : \mathbb{R} \to \mathbb{R}\) which converges uniformly to \(f : \mathbb{R} \to \mathbb{R}\). Prove that \(f\) is continuous on \(\mathbb{R}\).

4. Provide an example of a function \(f : [0, 1] \to \mathbb{R}\) which is not Riemann integrable. Prove that your \(f\) is not Riemann integrable.
1. Let $D$ be a space with the discrete topology containing at least two points. Prove that a topological space $X$ is connected if and only if each continuous function $f : X \to D$ is constant.

2. Prove that a finite union of compact subsets of a topological space is compact.

3. Suppose $X$ is a topological space, $Y$ is a compact Hausdorff space, and $f : X \to Y$ is a function.
   Proof: If $G = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$ with the usual product topology, then $f$ is continuous.
   Hint. $Y$ compact implies that the projection $\pi_1 : X \times Y \to X$ is a closed map.

4. Let $f : X \to Y$ be a continuous map from a compact space $X$ to a Hausdorff space $Y$. Prove the following.
   a. $f$ is a closed map.
   b. If $f$ is a bijection, then $f$ is a homeomorphism.
Applied Analysis

1. Solve the first order linear differential system

\[
X' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} X \quad \text{with initial condition} \quad X(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

2. Using power series find the solution about the point \( x_0 = 0 \) of the initial value problem. Your final solution should have at least eight nonzero terms. Also, determine the radius of convergence.

\[(x^2 + 4)y'' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2\]

3. a. Complete the “\( \varepsilon-N \)” definition: A sequence \( (a_n)_{n=0}^{\infty} \) converges to a limit \( L \) if . . . .

b. Use the \( \varepsilon-N \) definition to prove directly that

\[\text{if } a_n = \frac{2n^3 + 2}{3n^3 - 1}, \text{ then } (a_n)_{n=0}^{\infty} \text{ converges to } \frac{2}{3}.\]

4. Let the function \( f \) be differentiable at \( x = a \), and let \( p(x) \) be the tangent line to the graph of \( f \) at \( x = a \). Prove

\[\lim_{x \to a} \frac{|f(x) - p(x)|}{|x - a|} = 0.\]
1. Let \( g(x) = 2 \arctan x \).
   a. Prove that there exist exactly three solutions of the equation \( g(x) = x \).
   b. Let \( \alpha \) be the greatest of the three solutions. Find an interval \([a, b]\) such that for each \( x_0 \in [a, b] \), fixed-point iteration on \( g(x) \) will produce a sequence which converges to \( \alpha \). Justify your answer.

   Note: For this problem you may not use any graphing or rootfinding capabilities of your calculator.

2. Recall that if \( f \in C^2[a, b] \), the truncation error for the composite trapezoidal rule using \( n \) equal-sized intervals is given by
   \[
   E_{\text{Trap}} = -\frac{(b-a)^3}{12n^2} f''(\xi) \text{ for some } \xi \in (a, b).
   \]
   Consider approximating \( \int_0^1 e^{x^2} \, dx \) using the composite trapezoidal rule. Find an \( n \) so that by the error formula
   \[
   |E_{\text{Trap}_n}| < 10^{-6}.
   \]

3. Write an efficient algorithm to solve an \( n \times n \) linear system with four nonzero diagonals of the form \( Ax = y \) where
   \[
   A = \begin{pmatrix}
   a_1 & c_2 & d_3 \\
   b_1 & a_2 & c_3 & d_4 \\
   & b_2 & a_3 & c_4 & d_5 \\
   & & \ddots & \ddots & \ddots \\
   & & & b_{n-3} & a_{n-2} & c_{n-1} & d_n \\
   & & & b_{n-2} & a_{n-1} & c_n \\
   & & & & b_{n-1} & a_n
   \end{pmatrix}, \quad x = \begin{pmatrix}
   x_1 \\
   x_2 \\
   \vdots \\
   x_{n-2} \\
   x_{n-1} \\
   x_n
   \end{pmatrix}, \quad y = \begin{pmatrix}
   y_1 \\
   y_2 \\
   \vdots \\
   y_{n-2} \\
   y_{n-1} \\
   y_n
   \end{pmatrix}
   \]
   Assume that \( A \) is nonsingular and that pivoting is unnecessary. You should first convert to an upper triangular system and then apply back-substitution. Fully exploit the sparsity pattern.

4. Let \( \| \cdot \| \) be a vector norm on \( \mathbb{R}^n \) and define a matrix norm on the vector space of real \( n \times n \) matrices in the usual way by
   \[
   \|A\| = \max \{ \|Ax\| \mid \|x\| = 1 \}.
   \]
   Prove that \( \|Ax\| \leq \|A\| \|x\| \) for each \( A \) and for each \( x \in \mathbb{R}^n \).
1. Consider the following maximization problem.

maximize \[ 2x_1 + 4x_2 + 3x_3 + x_4 \]
subject to \[ 3x_1 + x_2 + x_3 + 4x_4 \leq 12 \]
\[ x_1 - 3x_2 + 2x_3 + 3x_4 \leq 7 \]
\[ 2x_1 + x_2 + 3x_3 - x_4 \leq 10 \]
\[ x_j \geq 0 \]

Using the Complementary Slackness Theorem, prove or disprove the following statements.
(a) \[ \left( \frac{26}{7}, 0, \frac{6}{7}, 0 \right) \] is an optimal solution.
(b) \[ (0, 10.4, 0, 0.4) \] is an optimal solution.

2. Consider the following maximization problem.

maximize \[ 3x_1 - 2x_2 + 4x_3 \]
subject to \[ 3x_1 + 2x_2 - 5x_3 \geq 15 \]
\[ 4x_1 + 2x_2 + x_3 \leq 18 \]
\[ 5x_1 - x_2 + 3x_3 \geq 10 \]
\[ x_1, x_2, x_3 \geq 0 \]

(a) Write the problem in canonical form.
(b) Use the Primal Simplex method to solve the problem.

3. Red Dwarf Toasters needs to produce 999 of their new “Talking Toaster”s. There are three ways this toaster can be produced: manually, semi-automatically, and robotically. Manual assembly requires 1 minute of skilled labor, 4 minutes of unskilled labor, and 3 minutes of assembly room time. The corresponding values for semiautomatic assembly are 4, 3, and 2; while those for robotic assembly are 8, 2, and 4. There are 4500 minutes of skilled labor, 3600 minutes of unskilled labor, and 2700 minutes of assembly room time available for this project. The total cost for producing manually is $7/toaster; semiautomatically is $8/toaster; and robotically is $8.50/toaster. Let \( x_1, x_2, \) and \( x_3 \) represent the number of toasters produced manually, semiautomatically, and robotically, respectively. The problem of producing 999 toasters at minimum cost is given (on the next page) as follows.
3. (Continued)

minimize \[ 7x_1 + 8x_2 + 8.5x_3 \]
subject to
\[ x_1 + 4x_2 + 8x_3 \leq 4500 \]
\[ 4x_1 + 3x_2 + 2x_3 \leq 3600 \]
\[ 3x_1 + 2x_2 + 4x_3 \leq 2700 \]
\[ x_1 + x_2 + x_3 = 999 \]
\[ x_1, x_2, x_3 \geq 0 \]

The beginning and final tableaux in the Simplex method are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_4 )</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>8</td>
<td>17</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer each question.

(a) What is the range on the cost of a robotically produced toaster, \( c_3 \), the coefficient of \( x_3 \), such that the basis variables do not change?

(b) Suppose that the number of minutes of unskilled labor available goes down from 3600 to 2900. How does that affect the solution?
There are four warehouses that have supplies of Product $P$ and four retail outlets that have demands for $P$. The boxed portion of the following table gives the unit costs of shipping Product $P$ from warehouses to retail outlets, with the supplies at each warehouse given along the column on the left of the box and the demands at each retail outlet along the row on top of the box. Use techniques of transportation problems to find the minimum total cost of shipping Product $P$ from the warehouses to meet the demands at the retail outlets.

\[
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
2 & 6 & 6 & 7 & 8 \\
2 & 2 & 4 & 2 & 1 \\
2 & 6 & 9 & 8 & 9 \\
2 & 1 & 3 & 2 & 3 \\
\end{array}
\]
1. Let $X$ represent the number of accidents per week that occur at the intersection of Mission Blvd. and Carlos Bee Blvd. Previous experience has shown that the mean of $X$ is 3.5. The times between accidents are independent of one another. That is to say, knowing that the last pair of accidents occurred six days apart does not give us any information about how much time will pass before the next accident.

(a) As specifically as you can, describe the distribution of $X$.
(b) What is the probability mass function of $X$?
(c) What is the standard deviation of $X$?
(d) Give the probability that there are fewer than two accidents this week at the intersection of Mission Blvd. and Carlos Bee Blvd.
(e) Give the probability that there are more than two accidents this week at the intersection of Mission Blvd. and Carlos Bee Blvd.
(f) Give the probability that there are two or more accidents this week at the intersection of Mission Blvd. and Carlos Bee Blvd.

2. Suppose that your wife is pregnant and due in 100 days from today. Suppose that the probability density function for having a child is approximately normal with mean 100 and standard deviation 8. You have a business trip and will return in 85 days from today and have to go on another business trip in 107 days from today.

(a) What is the probability that the birth will occur before your second trip?
(b) What is the probability that the birth will occur after you return from your first business trip?
(c) What is the probability that you will be there for the birth?
(d) Given that the birth does not occur before your first business trip, what is the probability that the birth will occur before your second trip?
(e) You are able to cancel your second business trip, and your boss tells you that you can return home from your first trip if there is a 99% chance that you will make it back for the birth. When must you return home?
3. Let $B_1, B_2, \ldots, B_n$ be the birthdays of $n$ independently chosen persons. Let $E_{i,j}$ be the indicator variable for the event that the $i$th and $j$th persons chosen have the same birthdays; i.e., the event $B_i = B_j$. For simplicity, we assume that for $i \neq j$ the probability of $B_i = B_j$ is $1/365$. So the $B_i$’s are mutually independent variables, and hence the $E_{i,j}$’s are pairwise independent variables.

Let $D$ be the number of matching pairs of birthdays among the $n$ choices, that is

$$D = \sum_{1 \leq i < j \leq n} E_{i,j}.$$ 

(a) Find the expected number of persons with matching pairs of birthdays, $E(D)$.

(b) Find the variance of $D$, $\text{Var}(D)$.

(c) Recall that Chebyshev’s theorem says that for any random variable $U$ with finite mean $\mu$ and finite variance $\sigma^2$ we have the following inequality.

$$P(|U - \mu| \geq k\sigma) \leq \left(1 - \frac{1}{k^2}\right)$$

Use your answers to (a) and (b) to give bounds on the number of pairs of persons with the same birthdays with better than 50% chance, assuming $n = 200$.

4. Suppose a population of forty-eight persons partitions as follows.

<table>
<thead>
<tr>
<th>Age</th>
<th>Young</th>
<th>Middle</th>
<th>Experienced</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>8</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>Female</td>
<td>8</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

The experiment is to draw twice from this population without replacement. You must show all work and reduce all fractions for full credit.

(a) What is the probability that both draws are “young” persons?

(b) What is the probability that the second draw is a “young” person?

(c) If the first draw is a male, what is the probability that the second draw is a “young” person?

(d) Are the events “draw a male on the first draw” and “draw a young person on the second draw” independent?