1. (a) State Cauchy’s integral formula for derivatives, and use it to derive Cauchy’s inequality.
   (b) State and prove Liouville’s theorem.

2. Let \( u(x, y) = x^3 - 4x^2 - 3xy^2 + 4y^2 \); show that \( u \) is harmonic, and find a harmonic conjugate function \( v(x, y) \) such that \( u + iv \) is entire.

3. Define the residue, \( \text{Res}(f(z), \infty) \), of the function \( f(z) \) at infinity, and use it to evaluate
   \[
   \int_C \frac{z^{10} + 4z^9 + 1}{z^2(z^8 + 1)} \, dz
   \]
   where \( C \) is the circle defined by \( |z - 2| = 4 \) traversed once counterclockwise.

4. Given \( f(z) = \frac{\sin^4(\pi z) \sinh^2(\pi z)}{(z^4 + 1)^5(z^2 - 36i)^2} \), find
   \[
   \int_C \frac{f'(z)}{f(z)} \, dz
   \]
   where \( C \) is the circle defined by \( |z| = \pi \) traversed once clockwise.
Real Analysis

1. Let $f : \mathbb{R} \to \mathbb{R}$.
   (a) Complete the $\varepsilon$-$\delta$ definition:
   \[
   \text{Definition. Given } x_0 \in \mathbb{R} \lim_{x \to x_0} f(x) = b \text{ if . . . .}
   \]
   (b) Use the definition to prove that $\lim_{x \to 2} (4x^2 - 1) = 15$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x - y|^2$ for each $x$ and $y$ in $\mathbb{R}$.
   (a) Prove that $f$ is differentiable in $\mathbb{R}$.
   (b) Prove that $f$ is a constant function.

3. Prove that if $0 < a < 5$, then
   \[
   \sum_{n=1}^{\infty} \frac{n(x - 3)^n}{5^{n+1}}
   \]
   converges uniformly on $[-2 + a, 8 - a]$.

4. (a) Complete the definition:
   \[
   A \subset \mathbb{R} \text{ is a set of measure } 0 \text{ if . . . .}
   \]
   (b) Prove that if $[a, b] \subset \mathbb{R}$, then each monotone increasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. 
1. Suppose $X$ is an infinite set, and $T = \{ A \subset X \mid A = \emptyset \text{ or } X - A \text{ is finite} \}$. Prove each of the following.
   (a) $T$ is a topology on $X$.
   (b) $(X, T)$ is connected.

2. Suppose $X$ is a compact topological space, $Y$ is a Hausdorff topological space, and $f : X \to Y$ is a continuous surjective map. Prove that for each subset $C \subset Y$, $C$ is closed in $Y$ if and only if $f^{-1}(C)$ is closed in $X$.

3. Suppose $X$ is a topological space, and $D$ is a dense subset of $X$. Prove that $\overline{D \cap U} = \overline{U}$ for each nonempty open set $U \subset X$.

4. Suppose $X$ is a normal space. Prove that if $A$ is a closed subset of $X$ and $U$ is an open set in $X$ such that $A \subset U$, then there exists an open set $V$ in $X$ including $A$ such that $\overline{V} \subset U$. 
1. The point $x_0 = 0$ is a regular point for the following ODE. Use power series to find the solution around $x_0 = 0$. You need to find only the first three terms of the series.

$$y' + \sin(x)y = 0, \quad y(0) = 6$$

2. Use the method of Fourier series to solve the following heat flow problem.

$$u_t = \alpha^2 u_{xx} \quad \forall x \in (0, \pi) \text{ and } t > 0$$

$$u(0, t) = 0$$

$$u(\pi, t) = 0$$

$$u(x, 0) = 100x$$

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$.

   (a) Complete the $\varepsilon$-$\delta$ definition:

   **Definition.** Given $x_0 \in \mathbb{R}$ \( \lim_{x \to x_0} f(x) = b \) if . . . .

   (b) Use the definition to prove that $\lim_{x \to 2} (4x^2 - 1) = 15$.

4. Prove that if $0 < a < 5$, then

$$\sum_{n=1}^{\infty} \frac{n(x - 3)^n}{5^n + 1}$$

converges uniformly on $[-2 + a, 8 - a]$. 

1. (a) Prove that there exist exactly two positive solutions of the equation \( x = 2 + \ln x \).
(b) Find an approximation \( \beta \) of the larger solution \( \alpha \) such that \( |\alpha - \beta| < 10^{-6} \).
(c) Prove that your approximation is in fact within \( 10^{-6} \) of (the exact) \( \alpha \).

Note: For this problem you may not use any graphing or root finding capabilities of your calculator.

2. Develop a \( O(h^2) \) formula for approximating \( f'(x_0) \) that uses \( f(x_0 - h), f(x_0), \) and \( f(x_0 + 2h) \) only.

3. For each of the systems (i) and (ii) of linear equations below determine for which the Jacobi or Gauss-Seidel iterative solution methods yield sequences of iterates which are certain to converge (for any initial approximation), and for which the iterative methods are certain to yield sequences of iterates which diverge (for any initial approximation except the actual exact solution). Give reasons, without actually carrying out the iterations.

\[
\begin{align*}
(i) & \quad 7x_1 + 2x_2 - x_3 = 12 \\
& \quad x_1 - 5x_2 + x_3 = -13 \\
& \quad x_1 + 4x_3 = 5 \\
(ii) & \quad x_1 - 5x_2 = -7 \\
& \quad 7x_1 - x_2 = 19 
\end{align*}
\]

4. Write an efficient algorithm to solve an \( n \times n \) linear system with four nonzero diagonals of the form \( Ax = y \) where

\[
A = \begin{pmatrix}
a_1 & d_2 & 0 & \ldots & 0 \\
b_1 & a_2 & d_3 & \ldots & 0 \\
c_1 & b_2 & a_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & c_{n-1} & b_{n-2} & a_{n-1} & d_n \\
0 & 0 & c_{n-2} & b_{n-2} & a_n \\
\end{pmatrix}, \quad x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n \\
\end{pmatrix}, \quad y = \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_{n-1} \\
y_n \\
\end{pmatrix}
\]

Assume that \( A \) is nonsingular and that pivoting is unnecessary. You should first convert to an upper-triangular system, and then apply back-substitution. Fully exploit the sparsity pattern.
1. Use the Primal Simplex method to solve the following minimization problem.

\[
\begin{align*}
\text{minimize} & \quad 20x_1 + 10x_2 + 12x_3 + 5 \\
\text{subject to} & \quad 3x_1 + 6x_2 + 4x_3 \geq 8 \\
& \quad -2x_1 + 6x_2 + 5x_3 \geq 3 \\
& \quad 4x_1 + 8x_2 + 10x_3 \leq 20 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

2. Consider the following maximization problem.

\[
\begin{align*}
\text{maximize} & \quad 2x_1 + 3x_2 - x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + x_3 \leq 6 \\
& \quad 4x_1 - x_2 + x_3 \leq 9 \\
& \quad 2x_1 + 3x_2 + 5x_3 \leq 20 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

The beginning and final tableaux in the Simplex method are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>4</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_6$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>1/3</td>
<td>4/9</td>
<td>-1/9</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>1/3</td>
<td>1/9</td>
<td>2/9</td>
<td>0</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>0</td>
<td>10/3</td>
<td>-14/9</td>
<td>-1/9</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>8/3</td>
<td>14/9</td>
<td>1/9</td>
<td>0</td>
</tr>
</tbody>
</table>

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer the questions.

(a) Suppose the coefficient $c_1$ of $x_1$ can be changed by a value of $\lambda$ and the coefficient $c_2$ of $x_2$ can be changed by a value of $3\lambda$. Give the range on $\lambda$, $c_1$, and $c_2$ such that the same basic variables are in the final solution.

(b) Suppose the right hand side constants in the three constraints change to 2, 17, and 10, respectively. Thus the constraints are now the following.

\[
\begin{align*}
\text{subject to} & \quad x_1 + 2x_2 + x_3 \leq 2 \\
& \quad 4x_1 - x_2 + x_3 \leq 17 \\
& \quad 2x_1 + 3x_2 + 5x_3 \leq 10 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Does the solution change? If so, find the new solution.
3. Using the Complementary Slackness Theorem prove or disprove the following statement. \([6, 8, 0]\) is the optimal solution of the maximization problem below.

\[
\begin{align*}
\text{maximize} & \quad 5x_1 + 3x_2 + 8x_3 \\
\text{subject to} & \quad 4x_1 - 2x_2 - x_3 \leq 9 \\
& \quad 3x_1 + x_2 + 2x_3 \leq 26 \\
& \quad -2x_1 + 2x_2 + 3x_3 \leq 4 \\
& \quad x_j \geq 0
\end{align*}
\]

4. A lumber company has three sources of wood and five markets where wood is demanded. The annual quantities of wood available in the three sources of supply are 15, 20, and 15 million board feet respectively. The amounts that can be sold at the five markets are 11, 12, 9, 10, and 8 million board feet, respectively. The company currently transports all of the wood by ship. It wishes to evaluate its transportation schedule, possibly shifting its transportation to rail. The unit cost of shipment (in $10,000) along the various routes using rail is described in the table below.

<table>
<thead>
<tr>
<th>Supply</th>
<th>Market 1</th>
<th>Market 2</th>
<th>Market 3</th>
<th>Market 4</th>
<th>Market 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>51</td>
<td>62</td>
<td>35</td>
<td>45</td>
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<td>20</td>
<td>59</td>
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<td>15</td>
<td>49</td>
<td>56</td>
<td>53</td>
<td>51</td>
<td>37</td>
</tr>
</tbody>
</table>

Determine the minimal cost for shipping all the wood by rail.
Department of Mathematics
Comprehensive Examination–Option III
2015 Autumn

Probability

1. For the following unrelated parts consider the following scenario. Nathan inadvertently mixed up six new batteries and four dead batteries in a drawer. When you are asked to find a probability calculate it to four decimal places.

   (a) If Nathan randomly selects one battery from the ten batteries, what is the probability that it is a dead battery?

   (b) If the ten batteries are all distinguishable, in how many different ways can Nathan put two batteries in a flashlight, four batteries in a radio, and four batteries in a camera?

   (c) If Nathan randomly selects four batteries from the ten batteries to install in a radio and the radio works only if none of the batteries is dead, what is the probability that the radio works?

   (d) If Nathan randomly selects a battery from the ten batteries and puts it in a toy pink bunny, and then randomly selects another battery from the remaining batteries and puts it in a second toy pink bunny, what is the probability that both bunnies operate? What is the probability that only one bunny operates? What is the probability that neither bunny operates?

   (e) Nathan randomly selects a battery from the ten batteries for his MP3 player, but is interrupted by a phone call. While he is answering the call his wife returns the battery to the drawer. After hanging up Nathan again selects a battery for his MP3 player. What is the probability that at least one of his two selections results in a dead battery?

2. Let $Y$ be a continuous random variable with the following probability density function (pdf).

   $$f(y) = \begin{cases} 2(1 - y) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $f(y)$ is a pdf.

   (b) Find $P(0.10 < Y < 0.5)$.

   (c) Let $X = Y^2$. Find the probability density function for $X$.

   (d) Find $P(X > 0.49)$.

   (e) Find $E(Y)$.

   (f) Find $E(X)$. 
3. Let $X$ denote the weight (in tons) of a bulk item supplied at the beginning of the week and assume that $X$ is uniform on the interval $(0.5, 1.5)$. Let $Y$ denote the weight (in tons) of the bulk item sold over the week (sold only from the amount supplied at the beginning of the week) and assume that $Y$ is uniform on the interval $(0, x)$ where $x =$ the weight delivered at the beginning of the week.

(a) If the amount supplied at the beginning of the week is one ton, what amount could be expected to sell that week?
(b) Derive a general expression for $E(Y - X = x)$.
(c) If the amount supplied at the beginning of the week is one ton, what is the variance for the amount sold that week?
(d) Derive a general expression for the variance of $Y$ given that $X = x$.
(e) Find the unconditional expectation of $Y$.
(f) Find the unconditional variation of $Y$.

4. You have one fair coin and one biased coin which lands Heads with probability $3/4$. You pick one of the coins at random and flip it three times. It lands Heads all three times.

Let $A$ be the event that the chosen coin lands Heads three times. Let $F$ be the event that you picked a fair coin.

(a) Compute the conditional probability that the chosen coin lands Heads three times, given the selected coin is fair. Compute $P(A|F)$.
(b) Compute the conditional probability that the chosen coin lands Heads three times, given the selected coin is not fair. Compute $P(A|F^c)$.
(c) Compute the probability of the chosen coin landing Heads three times. Compute $P(A)$.
(d) You are interested in the conditional probability that the coin is fair, given that the selected coin landed Heads three times. Compute $P(F|A)$. 