Department of Mathematics
Comprehensive Examination - Option I
2019 Spring Semester

Topology

1. Let A and B be subsets of a topological space X. Prove: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. Let $f{:}X \to Y$ be a continuous bijective function from a compact space X to a Hausdorff space Y. Prove that f is a homeomorphism.

3. Let D be a space with discrete topology having at least two points. Prove: A topological space X is connected if and only if every continuous function $f{:}X \to D$ is constant.

4. Let X and Y be topological spaces and

$$\mathcal{C} = \{U \times V{:}U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

   (a) Prove: $\mathcal{C}$ forms a basis for a topology, the product topology, on $X \times Y$.

   (b) Let $\pi : X \times Y \to X$ be the projection map defined by $\pi(x, y) = x \; \forall \; (x, y) \in X \times Y$. Use the tube lemma to prove that if Y is compact, then π is a closed map, i.e., $\pi(C)$ is closed in X for every closed set C in $X \times Y$.

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Algebra

1. Let G be a group and H be the subgroup generated by $\{x^{-1}y^{-1}xy \mid x, y \in G\}$. Prove:
   (a) H is normal in G.
   (b) G/H is abelian.

2. Let $\mathbf{Z}$ be the ring of integers and suppose $\phi : \mathbf{Z} \to R$ is a homomorphism of $\mathbf{Z}$ onto the ring R. Prove that R is isomorphic as a ring either to $\mathbf{Z}_n$ for some positive integer n, or to $\mathbf{Z}$.

3. Prove that each finite integral domain is a field; deduce that for each prime $p, \mathbf{Z}_p$ is a field.

4. Let V and W be vector spaces over a field, and suppose $T{:}V \to W$ is a linear transformation. Prove:
   (a) $N(T) = \{\mathbf{x} \in V \mid T(\mathbf{x}) = \mathbf{0}\}$ is a subspace of V.
   (b) T is one-to-one if and only if $N(T) = \{\mathbf{0}\}$.
   (c) If for each linearly independent subset S of V, $T(S)$ is a linearly independent subset of W, then T is one-to-one.

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Numerical Analysis

1. Let $g(x) = 4\ln(x)$.

   (a) Prove that the equation $x = g(x)$ has exactly two positive solutions.

   (b) Consider approximating the larger solution, α, by applying a fixed-point iteration on $g(x)$. Determine an interval $[a, b]$ such that the iteration sequence will converge to α for any initial approximation $x_0 \in [a, b]$. Justify your answer.

2. Suppose that $f^{(5)}$ is continuous. Use Taylor's Theorem to show that

$$f'''(x_0) \approx \frac{-f(x_0 - 2h) + 2f(x_0 - h) - 2f(x_0 + h) + f(x_0 + 2h)}{2h^3} \text{ is } \mathcal{O}(h^2)$$

3. Let A be an $n \times n$ system of the form

$$\begin{bmatrix} a_1 & d_2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ b_1 & a_2 & d_3 & 0 & & & & & & \cdot \\ c_1 & b_2 & a_3 & d_4 & 0 & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & 0 & c_{n-3} & b_{n-2} & a_{n-1} & d_n \\ 0 & & & & & & 0 & c_{n-2} & b_{n-1} & a_n \end{bmatrix}$$

   Write an algorithm that generates the upper-triangular result of applying Gaussian Elimination to A. That is, construct the matrix U with just 2 significant diagonal vectors so that

$$U = \begin{bmatrix} a_1 & d_2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_2 & d_3 & 0 & & & & \cdot \\ \cdot & 0 & a_3 & d_4 & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & 0 & a_{n-1} & d_n \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_n \end{bmatrix}$$

   You may assume that pivoting in unnecessary. Exploit the sparsity pattern by writing everything in terms of the given vectors. Do not use double subscripts. As in the general algorithm, we don't care what happens to the values in the b and c vectors. It's only necessary that the vectors a and d wind up with their appropriate values.

4. (a) State the Gershgorin Circle Theorem.

   (b) Use the Gershgorin Circle Theorem to prove that each strictly diagonally dominant real $n \times n$ matrix is nonsingular.

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Applied Analysis

1. Solve the linear system $X' = AX$ with

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & 2 \\ 6 & -6 & 5 \end{pmatrix} \text{ and initial condition } X(0) = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

2. Find the first four nonzero terms of two linearly independent solutions to the following differential equation about the point $x = 0$, using power series solutions.

$$(x-1)y'' + 3xy' - y = 0$$

3. Prove:

   (a) If $f : [0,1] \to [0,1]$ is continuous, then there exists a fixed point of f.

   (b) There exists no continuous function mapping the closed interval $[0,1]$ onto the open interval $(0,1)$.

4. Suppose that $0 < a < \frac{5}{3}$. Prove that

$$\sum_{n=0}^{\infty} \frac{nx^n 3^n}{5^{n+2}}$$

is uniformly convergent on $[-a, a]$.

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Linear Programming

1. Solve the following minimization problem using the primal simplex method.

$$\begin{array}{rrl} \text{minimize} & 6x_1 + 6x_2 + 5x_3 & \\ \text{subject to} & -5x_1 + 4x_2 + x_3 & \leq 18 \\ & 2x_1 - x_2 + 2x_3 & \geq 10 \\ & 3x_1 + 3x_2 + 4x_3 & \geq 12 \\ & x_1, x_2, x_3 & \geq 0 \end{array}$$

2. Consider the following problem.

$$\begin{array}{rrrrl} \text{minimize} & -3x_1 & -13x_2 & -13x_3 & \\ \text{subject to} & x_1 & +\ x_2 & & \leq 7 \\ & x_1 & -3x_2 & +2x_3 & \leq 15 \\ & & 2x_2 & +3x_3 & \leq 9 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

The initial and final tableaux are

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	0	1	0	0	7
x_5	1	3	2	0	1	0	15
x_6	0	2	3	0	0	1	9
	-3	-13	-13	0	0	0	0
x_3	0	0	1	1	-1	1	1
x_1	1	0	0	$\frac{5}{2}$	$-\frac{3}{2}$	1	4
x_2	0	1	0	$-\frac{3}{2}$	$\frac{3}{2}$	-1	3
	0	0	0	1	2	3	64

Solve the following using sensitivity analysis. For each situation return to the original problem statement.

(a) Find the range that the value of $b_1 = 7$ can have without changing the optimal solution. That is, same basic variables as the original problem, although their final values will be altered.

(b) Add in the following constraint to the original problem and give the new solution.

$$x_1 + 3x_2 \leq 7$$

(c) What is the range on the coefficient, c_2, of x_2 in the objective function of the original problem that still maintains the same optimal solution of x values?

3. Using the Complementary Slackness Theorem, determine whether $(4, 0, 5)$ is the optimal solution to

$$\begin{array}{rrrrl} \text{maximize} & 6x_1 & +2x_2 & +7x_3 & \\ \text{subject to} & 3x_1 & +2x_2 & +3x_3 & \leq 30 \\ & 2x_1 & -4x_2 & +8x_3 & \leq 48 \\ & 5x_1 & +2x_2 & +\ x_3 & \leq 25 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

4. A manufacturer of tribbles has warehouses in Atlanta, Baltimore, Chicago, and Detroit. The warehouse in Atlanta has 40 tribbles in stock, the warehouse in Baltimore has 70 tribbles in stock, the warehouse in Chicago has 40 tribbles in stock, and the warehouse in Detroit has 30 tribbles in stock. There are retail stores in Eugene, Fairview, Grove City, and Hayward that need to receive 50, 60, 30, and 40 tribbles, respectively. The following table gives the unit shipping costs from each warehouse to each retail store.

	Eugene	**Fairview**	**Grove City**	**Hayward**
Atlanta	10	15	20	18
Baltimore	20	22	18	15
Chicago	19	20	21	18
Detroit	25	30	28	29

Determine a shipping program that fulfills the required demands and minimizes the total shipping cost.

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Probability

1. A salesperson has three clients in district A, five clients in district B, and seven clients in district C.

   (a) In how many ways can the salesperson select a group of clients so that in this group there are exactly two clients from each district?

   (b) In how many ways can the salesperson select six clients from these districts?

   (c) If the salesperson randomly selects six clients from these districts, then what is the probability that at least one of them is from district A?

   (d) If the salesperson randomly selects six clients from these districts and none of them is from district B, then what is the probability that exactly two of them are from district A?

   (e) If the salesperson randomly selects six clients from these districts and exactly three of them are from district B, then what is the probability that the salesperson selects the other three clients from district C?

2. Suppose that the cost X, in dollars, and the time Y, in minutes, of a job have the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{5000} & \text{if } 100 > x > y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Let g be the probability density function of X and let h be the probability density function of Y. Find g and h.

   (b) Are X and Y independent? Prove your answer.

   (c) Find $E(XY)$.

   (d) Find $P[X > 2Y]$.

   (e) Find the probability density function of $Z = X - Y$.

3. A binary communications system consists of a transmitter that sends 0s and 1s to a receiver over a communication channel. Sometimes errors occur, so that when a 1 is sent a 0 is received, and *vice-versa*. The probabilities of sendings 0s and 1s are p_0 and p_1, respectively. The probability of receiving a 0 when a 1 is sent and receiving a 1 when a 0 is sent is p.

   (a) Develop an expression for $P(R_1)$, the probability of receiving 1s.

   (b) Develop an expression for $P(R_0)$, the probability of receiving 0s.

   (c) Find an expression for $P(S_1|R_1)$; i.e. given 1 is received, what is the probability that 1 is sent?

   (d) Find an expression for $P(S_0|R_0)$; i.e. given 0 is received, what is the probability that 0 is sent?

   (e) Find an expression for the probability of an error in the system.

4. The amount of bread (in hundreds of pounds) that a bakery is able to sell in one day is found to be a real-valued random variable X, with a probability density function given by

$$f(x) = \begin{cases} cx & \text{if } 0 \leq x < 5 \\ c(10-x) & \text{if } 5 \leq x < 10 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the value of c that makes $f(x)$ a density function.

   (b) Graph the density function.

   (c) What is the probability that the number of pounds of bread that will be sold tomorrow is

      i. more than 500 pounds?

      ii. less than 500 pounds?

      iii. between 250 pounds and 750 pounds?

   (d) Let A be the event that more than 500 pounds of bread is sold in a day; let B be the event that less than 500 pounds of bread is sold in a day; and let C be the event that between 250 pounds and 750 pounds of bread is sold in a day.

      i. Find $P(A|B)$.

      ii. Find $P(A|C)$.

      iii. Are events A and B independent? Justify your answer.

      iv. Are events A and C independent? Justify your answer.